Chapter 1

What Is Discrete Mathematics?

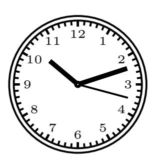
It is barely an oversimplification to say that discrete mathematics is the study of structures that can be typed on a computer keyboard.

To begin to understand what this means, it is helpful to appreciate the meanings of (and differences between) analog and discrete systems. Discrete mathematics is the study of *discrete* (as opposed to *analog*) systems.

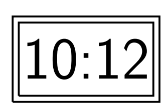
1.1 Analog Versus Discrete Systems

The difference between analog and discrete systems is that analog systems involve smooth, continuous, unbroken movement or structures, whereas discrete systems involve individual parts or states that are clearly separate from one another.

This is illustrated by the difference between a traditional (analog) clock and a digital (discrete) clock, two inherently different systems for recording time. The hands of an analog clock move in a fluid, continuous motion. In the one minute between 10:12 and 10:13, the minute hand moves in a smooth, unbroken motion passing through all instants between these two times. In an hour it passes through infinitely many different instants of time. This is an analog system. By contrast, a digital clock jumps from 10:12 to 10:13 in an instant. In an hour it records a finite number (in fact, 60) instants of time. This is a *discrete* system.



Analog clock

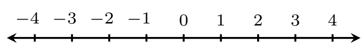


Discrete (digital) clock

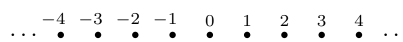
Think of it this way. Regarding the analog clock, you could never type out the infinitely many instants of time between, say, 10:00 and 11:00 am. But you can do this for the digital clock, as 10:00, 10:01, 10:02, … 10:59, 11:00, just using the numeric symbols on the keypad, along with the colon.

Discrete mathematics deals with structures and systems that can be typed.

For another example of an analog versus a discrete system, consider real numbers versus integers. We visualize the real numbers as a smooth, unbroken, infinitely long line. You can put your finger on 0 and move it continuously to the right in a fluid motion, stopping at (say) 3. As you do this, your finger moves through infinitely many numbers, one for each point on the line from 0 to 3. This is an analog system.



Real numbers (analog system)



Integers (discrete system)

By contrast, the integers (whole numbers) $\ldots -3, -2, -1, 0, 1, 2, 3 \ldots$ are a system whose parts (numbers in this case) are discrete entities. Putting your finger at 0 and moving to the right, you jump from 0, to 1, to 2, to 3, and so on.

Think of it this way. The set of integers is a discrete system because any integer, such as, say 243 and -11 can be expressed by typing a finite sequence of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and possibly a minus sign "$-$". But the system of real numbers is **not** a discrete system: it has irrational numbers like $\pi = 3.14159265359\ldots$ that cannot be typed because they involve infinitely many non-repeating decimal places. *Discrete mathematics deals with structures and systems that can be typed.*

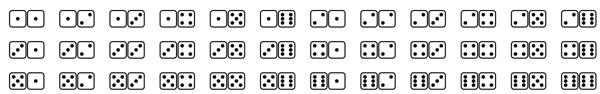
In this vein, the set of rational numbers (fractions of integers) is also a discrete system because any rational number, such as $-397/24$ is expressible as a finite sequence of the symbols $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, -$ and /, and thus can be typed. By contrast, the (analog) system of real numbers includes irrational numbers like π and $\sqrt{2}$ (just to name two) that cannot be expressed as fractions of integers.

The focus of discrete mathematics is on discrete number systems like the integers or the rational numbers (as well as many other discrete number systems that we will encounter). This is not to say that the real number system is unimportant. The powerful theory of calculus is built on the real number system. It is the language of physics, and it can also be a useful tool in discrete mathematics. Although calculus will not be used in this text (which is a first course in discrete mathematics), if you go on to become a serious practitioner of discrete mathematics, then calculus will be a part of your mathematical tool box. But even then—as now—your primary focus will be on discrete systems.

1.2 Examples of Discrete Structures

We just considered three examples of discrete systems, or structures: digital timekeeping, the integers and the rational numbers. We now sample some other types of discrete structures. Our list will be intentionally brief and (by necessity) incomplete. You will see many, many other types of discrete structures in this course, but these examples should give you the flavor of what is to come.

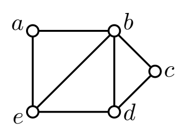
For the first example, consider the process of rolling a dice two times in a row. A typical outcome might be described by a pair , meaning that a 3 was rolled first, followed by a 5. You could also describe this outcome with the ordered pair $(3, 5)$. There are exactly 36 outcomes, listed below.



This is a discrete system because its 36 parts are distinct and separate from one another. (And moreover, each can be fully described with just two symbols. Each can be *typed*.)

The set of all such outcomes is called a *sample space*. Such sets can be useful. For example, if we wanted to know the probability of rolling a double, we can see that exactly 6 of these 36 equally likely outcomes is a double, so the probability of rolling a double is $\frac{6}{36} = \frac{1}{6}$. Chapters 2 and 5 introduce sets and counting, theories relevant to situations such as this, and we study probability in Chapter 6.

Another example of a discrete structure is a *graph*. In mathematics, the word "graph" is used in two different contexts. In algebra or calculus a *graph* is a visual description of a function, graphed on a coordinate axis. Although we do use such graphs in discrete mathematics, we more often use the word *graph* to mean a network of nodes with connections between them. Here is a picture of a typical graph.



Its nodes are described by the discrete set $\{a, b, c, d, e\}$, and its connections (called *edges*) are $\{ab, bc, cd, de, ea, eb\}$. Therefore this particular graph is completely described by typing the information

$$\Big(\{a, b, c, d, e\}, \quad \{ab, bc, cd, de, ea, eb\}\Big).$$

The theory of graphs is a major branch of discrete mathematics. Graphs have wide-ranging applications. For example, the Internet is a huge graph whose nodes are web pages and whose edges are links between them. Google's search algorithm involves the mathematics of this structure. Also, a network of roads is a graph, and finding the shortest route between nodes is one example of a typical discrete mathematics problem. We will study graphs in Chapter 15.

A third example of a discrete structure is a computer program. A program is a file of characters (that can be typed on a computer keyboard!) consisting of a sequence of commands that can be executed by a digital computer. As you will see in this course, computer programs (more generically known as *algorithms*) are within the scope of discrete mathematics too.

1.3 Symbols

Symbols play a big role in discrete mathematics. We will use a great many symbols. You can think of them as being the characters on a very extensive computer keyboard. This will include the upper- and lower-case letters of the English alphabet (A, B, C, ..., Z, a, b, c, ..., z) as well as the digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and special characters (+, =, /, #, $, %, &, ∗, etc.). Imagine that the keyboard also features greek letters ($\alpha, \beta, \gamma, \ldots, \Delta, \Gamma, \Sigma, \Upsilon \ldots$). It will be convenient to also use symbols like ♡, ♣, ♢, ♠, ⚀, ⚁, ⚂, etc. as well as some others that will be introduced in due time. Mathematics is case-sensitive, so, for example, "A" and "a" are two different symbols.

We often attach special meanings to symbols. For example, the symbol "5" typically stands for the number five. But there is a difference between the symbol and what it stands for, just as there is a difference between your name and *you*. The number five can be designated by the Hindu-Arabic symbol "5" or the Roman symbols "V" or "v". The ancient Babylonians use the cuneiform symbol 𒐊 for five. The symbols 5, V, v and 𒐊, are just symbols. We understand them to be different *names* for the numeric value five. But they are *symbols*, not numbers.

Strings of multiple symbols can stand for various things. For example, the string of symbols 1040 stands for the number one thousand forty. Each digit in this string carries a specific meaning. For another example, the words on this page are strings of letters and (for the most part) the individual letters stand for specific sounds.

By contrast, consider a password, such as X1040f$. By themselves, the individual symbols in this password have absolutely no meaning. They are just individual pieces of a larger object whose meaning comes only from the aggregate. Likewise, consider the graph on page 5, which was described by the string of symbols $\big(\{a, b, c, d, e\},\ \{ab, bc, cd, de, ea, eb\}\big)$. The letters in this string of symbols don't have any meaning other than being names for the nodes. This graph could just as validly be described by a string such as $\big(\{\alpha, 6, c, \Delta, e\},\ \{\alpha 6, 6c, c\Delta, \Delta e, e\alpha, e6\}\big)$, where the a is replaced with α, the b with 6, and the d with Δ. One interesting feature of discrete mathematics is that it is often concerned with meaningful structures made up of symbols that are not assigned meanings of their own.

Another very important type of symbol (which you encountered in algebra) is a *variable*. A **variable** is a symbol that stands for a definite but unspecified thing. We often reserve the lower-case symbols s, t, u, v, w, x, y, z at the end of the alphabet for variables. A variable may stand for a numeric value in an equation or formula. For instance, the variable x in the equation $x^2 - 3x + 2 = 0$ stands for a number that makes the equation true. There are two such numbers, namely $x = 1$ and $x = 2$.

For another example, consider the equation $y = 2x^2 - 4$ containing variables x and y. This expresses a relationship between the quantities x and y. If $x = 1$, then $y = -3$; if $x = 2$, then $y = 0$, etc.

As you progress through this course you will become accustomed to variables that stand for things more complex than numbers, like $x =$ ♣ or $x = \{$⚀⚂, ⚁⚁, ⚂⚀$\}$.

1.4 Prerequsites

It is understood that you come to discrete mathematics with a sound understanding of arithmetic and algebra. For instance, you can add (or subtract) fractions by getting a common denominator, as in

$$\frac{5}{3}+\frac{3}{2} \;=\; \frac{5}{3}\cdot\frac{2}{2}+\frac{3}{2}\cdot\frac{3}{3} \;=\; \frac{10}{6}+\frac{9}{6} \;=\; \frac{19}{6},$$

and you can do this without a calculator. You can also carry out such operations with expressions having variables, and perform the consequent simplifications, as in

$$\frac{x(x+1)}{2}-x \;=\; \frac{x(x+1)}{2}-\frac{2x}{2} \;=\; \frac{x(x-1)-2x}{2} \;=\; \frac{x^2-3x}{2}.$$

You know that $(x+2)^2 \neq x^2+4$, but rather $(x+2)^2 = (x+2)(x+2) = x^2+4x+4$ (for example). You can also compute, say, $(x+2)^3$ as

$$(x+2)^3 = (x+2)(x+2)^2 = (x+2)(x^2+4x+4) = x^3+6x^2+12x+8.$$

You have a working knowledge of various exponential laws such as $a^n a^m = a^{m+n}$, $(a^m)^n = a^{mn}$ and $(ab)^n = a^n b^n$. For example, you understand why

$$\frac{3^{n+1}}{3} \;=\; \frac{3^1 3^n}{3} = 3^n \quad \text{and} \quad \frac{\left(3x^5\right)^2}{6x} \;=\; \frac{3^2 x^{10}}{6x} \;=\; \frac{3x^9}{2}.$$

If your arithmetic and algebra skills are rusty, then it is a prerequisite that you are willing to take any necessary steps to overcome any and all deficiencies. You can do this before starting the course, or as you go along. Working competently with algebraic expressions will be essential for progress in discrete mathematics.

Although calculus is not necessary to understand the ideas in this book, you have probably studied it. If so, that background will serve you well. For instance, calculus requires a certain fluency in algebra and arithmetic, and that fluency is equally essential in discrete mathematics, as we noted above. Calculus also requires a working knowledge of functions, and that background will be useful. It has also given you a grounding in certain useful notations, such as the sigma notation for expressing sums. Given a list of numbers $a_1, a_2, a_3, \ldots, a_n$, their sum is compactly expressed as

$$a_1+a_2+a_3+\cdots+a_n = \sum_{k=1}^{n} a_k.$$

All of these background topics will play a role for us.

Finally, one very important prerequisite is that you are willing to think about things in new ways, to work hard, to make mistakes and learn from them, and to persevere.

1.5 Case Study: Binary and Hexadecimal Number Systems

The binary and hexadecimal number systems are good illustrations of an adaptive use of symbols. Though not absolutely essential for much of this text, it is important because it forms the basis for the internal workings of computer circuitry and computations.

We will introduce binary numbers by first reviewing the familiar decimal system. Exponents play a key role here. Recall that for any number a and positive integer n, the power $a^n = a \cdot a \cdot \cdots \cdot a$ is the product of a with itself n times. For instance, $10^3 = 1000$. Recall from algebra that if a is non-zero, then $a^0 = 1$. In the following pages you will encounter $10^0 = 1$, $2^0 = 1$ and $16^0 = 1$.

In daily life we use the familiar base-10 number system, also called the **Hindu-Arabic** number system, or the **decimal** number system. It uses ten symbols $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, representing the quantities zero through nine. There is no single symbol for the quantity ten – instead we express it as the combination "10," signifying that ten equals 1 ten plus 0 ones. Any other positive integer is represented as a string of symbols $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, standing for the sum of the digits in the string times powers of ten, decreasing to the zeroth power $10^0 = 1$. For example,

$$\begin{aligned} 7406 &= \mathbf{7} \cdot 10^3 \ + \ \mathbf{4} \cdot 10^2 \ + \ \mathbf{0} \cdot 10^1 + \mathbf{6} \cdot 10^0 \\ &= \mathbf{7} \cdot 1000 + \ \mathbf{4} \cdot 100 \ + \mathbf{0} \cdot 10 \ + \ \mathbf{6} \cdot 1. \end{aligned}$$

Thus the number seven-thousand-four-hundred-six is represented as 7406, with a 7 in the *thousand's place*, a 4 in the *hundred's place*, a 0 in the *ten's place*, and a 6 in the *one's place*. There is little need to elaborate because you internalized this early in life.

There is nothing sacred about base of ten, other than the fact that it caters to humans (who have ten fingers). The **base-2**, or **binary** number system, which we now introduce, expresses numbers in terms of powers of 2 rather than powers of 10.

The binary number system uses only two digits, 0 and 1, representing the quantities zero and one. There is no single symbol for the number two – instead it is expressed as the combination "10," signifying that two equals 1 two plus 0 ones. Any other quantity is represented as a string of 0's and 1's, such as 10011. Such a string stands for the number that equals the sum of the digits in the string times powers of two, decreasing to $2^0 = 1$.

For example, the base-2 number **10011** equals the base-10 number

$$\begin{gathered} \mathbf{1} \cdot 2^4 \ + \ \mathbf{0} \cdot 2^3 + \ \mathbf{0} \cdot 2^2 + \ \mathbf{1} \cdot 2^1 + \ \mathbf{1} \cdot 2^0 = \\ \mathbf{1} \cdot 16 \ + \ \mathbf{0} \cdot 8 \ + \ \mathbf{0} \cdot 4 \ + \ \mathbf{1} \cdot 2 \ + \ \mathbf{1} \cdot 1 \ \ = 19. \end{gathered}$$

The number nineteen is represented as "10011" in base-2 because it is the sum of **1** sixteen, **0** eights, **0** fours, **1** two and **1** one. It is represented as "19" in base-10 because it is the sum of **1** ten and **9** ones.

For clarity, we sometimes use a subscript to indicate what base is being used, so the above computation is summarized as $10011_2 = 19_{10}$.

Table 1.1 shows the first sixteen binary numbers in the left column, with their corresponding decimal representations on the right. Be sure you agree with this. For instance, $110_2 = 6_{10}$ because $\mathbf{1} \cdot 2^2 + \mathbf{1} \cdot 2^1 + \mathbf{0} \cdot 2^0 = 6$.

Table 1.1 Binary and decimal representations of numbers

binary number		powers of 2 (16 8 4 2 1)		decimal number
0	=	$\mathbf{0}\cdot 1$	=	0
1	=	$\mathbf{1}\cdot 1$	=	1
10	=	$\mathbf{1}\cdot 2+\mathbf{0}\cdot 1$	=	2
11	=	$\mathbf{1}\cdot 2+\mathbf{1}\cdot 1$	=	3
100	=	$\mathbf{1}\cdot 4+\mathbf{0}\cdot 2+\mathbf{0}\cdot 1$	=	4
101	=	$\mathbf{1}\cdot 4+\mathbf{0}\cdot 2+\mathbf{1}\cdot 1$	=	5
110	=	$\mathbf{1}\cdot 4+\mathbf{1}\cdot 2+\mathbf{0}\cdot 1$	=	6
111	=	$\mathbf{1}\cdot 4+\mathbf{1}\cdot 2+\mathbf{1}\cdot 1$	=	7
1000	=	$\mathbf{1}\cdot 8+\mathbf{0}\cdot 4+\mathbf{0}\cdot 2+\mathbf{0}\cdot 1$	=	8
1001	=	$\mathbf{1}\cdot 8+\mathbf{0}\cdot 4+\mathbf{0}\cdot 2+\mathbf{1}\cdot 1$	=	9
1010	=	$\mathbf{1}\cdot 8+\mathbf{0}\cdot 4+\mathbf{1}\cdot 2+\mathbf{0}\cdot 1$	=	10
1011	=	$\mathbf{1}\cdot 8+\mathbf{0}\cdot 4+\mathbf{1}\cdot 2+\mathbf{1}\cdot 1$	=	11
1100	=	$\mathbf{1}\cdot 8+\mathbf{1}\cdot 4+\mathbf{0}\cdot 2+\mathbf{0}\cdot 1$	=	12
1101	=	$\mathbf{1}\cdot 8+\mathbf{1}\cdot 4+\mathbf{0}\cdot 2+\mathbf{1}\cdot 1$	=	13
1110	=	$\mathbf{1}\cdot 8+\mathbf{1}\cdot 4+\mathbf{1}\cdot 2+\mathbf{0}\cdot 1$	=	14
1111	=	$\mathbf{1}\cdot 8+\mathbf{1}\cdot 4+\mathbf{1}\cdot 2+\mathbf{1}\cdot 1$	=	15
10000	=	$\mathbf{1}\cdot 16+\mathbf{0}\cdot 8+\mathbf{0}\cdot 4+\mathbf{0}\cdot 2+\mathbf{0}\cdot 1$	=	16

In converting between binary and decimal representations of numbers, it's helpful to know the various powers of 2. They are listed in Table 1.2. For example, $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$, so 32 appears below 2^5.

Table 1.2 Powers of 2

$\cdots$	2^{12}	2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
$\cdots$	4096	2048	1024	512	256	128	64	32	16	8	4	2	1

Table 1.1 suggests that to convert a given a binary number to decimal, multiply its digits by decreasing powers of two, down to $2^0 = 1$, and add them. For example,

$$\begin{aligned} 1110 &= \mathbf{1} \cdot 2^3 + \mathbf{1} \cdot 2^2 + \mathbf{1} \cdot 2^1 + \mathbf{0} \cdot 2^0 \\ &= \mathbf{1} \cdot 8 \ + \ \mathbf{1} \cdot 4 \ + \ \mathbf{1} \cdot 2 \ + \ \mathbf{0} \cdot 1 = 14. \end{aligned}$$

Example 1.1. Convert the binary number 110101 to decimal.

Solution We simply write this number as a sum of powers of 2 in base-10.

$$\begin{aligned} 110101 &= \mathbf{1} \cdot 2^5 + \mathbf{1} \cdot 2^4 + \mathbf{0} \cdot 2^3 + \mathbf{1} \cdot 2^2 + \mathbf{0} \cdot 2^1 + \mathbf{1} \cdot 2^0 \\ &= \mathbf{1} \cdot 32 + \mathbf{1} \cdot 16 + \mathbf{0} \cdot 8 + \mathbf{1} \cdot 4 + \mathbf{0} \cdot 2 + \mathbf{1} \cdot 1 = 53 \end{aligned}$$

Thus $110101_2 = 53_{10}$, that is, 110101 (base 2) is 53 (base 10). ✍

Converting decimal to binary involves running this process in reverse, which can involve some reverse engineering.

Example 1.2. Convert the decimal number 347 to binary.

Solution We need to find how 347 is a sum of powers of 2. Table 1.2 shows that the highest power of 2 less than 347 is $2^8 = 256$, and

$$\begin{aligned} 347 &= 256 + 91 \\ &= \ 2^8 \ + \ 91. \end{aligned}$$

Now look at the 91. Table 1.2 shows that the highest power of 2 less than 91 is $2^6 = 64$, and $91 = 64 + 27 = 2^6 + 27$, so the above becomes

$$347 = 2^8 + 2^6 + 27.$$

From here we can reason out $27 = 16 + 8 + 2 + 1 = 2^4 + 2^3 + 2^1 + 2^0$. Therefore

$$347 = 2^8 + 2^6 + 2^4 + 2^3 + 2^1 + 2^0.$$

Powers 2^7, 2^5 and 2^2 do not appear, so we insert them, multiplied by 0:

$$347 = \mathbf{1} \cdot 2^8 + \mathbf{0} \cdot 2^7 + \mathbf{1} \cdot 2^6 + \mathbf{0} \cdot 2^5 + \mathbf{1} \cdot 2^4 + \mathbf{1} \cdot 2^3 + \mathbf{0} \cdot 2^2 + \mathbf{1} \cdot 2^1 + \mathbf{1} \cdot 2^0.$$

Therefore 347 is the base-2 number 101011011. ✍

For any integer $n > 1$ there is a base-n number system that uses n symbols. Various cultures throughout history have used different base-n number systems. The ancient Babylonians used a base-60 system with 60 different cuneiform digits (including a blank, used for what we now call 0). The Aztecs used base-20. In the modern era, some early computers used the base-3 system, with three digits represented by a positive, zero or negative voltage.

Today the binary system is the foundation for computer circuitry, with 0 represented by a zero voltage, and 1 by a positive voltage. Though the binary system has just two digits, it is inefficient in the sense that many digits are needed to express even relatively small numbers. The base-16, or hexadecimal system, which we study next, remedies this. It is closely related to binary, but it is much more compact.

Hexadecimal Numbers

Base-16 is called the **hexadecimal** number system. It uses 16 symbols, including the familiar ten symbols $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ representing the numbers zero through nine, plus the six additional symbols A, B, C, D, E, and F, representing the numbers ten through fifteen.

Table 1.3 summarizes this. It shows the numbers zero through fifteen in decimal, binary and hexadecimal notation. For consistency we have represented all binary numbers as 4-digit strings of 0's and 1's by adding zeros to the left, where needed.

Table 1.3 The first sixteen integers in decimal, binary and hexadecimal

decimal	binary	hexadecimal
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
8	1000	8
9	1001	9
10	1010	A
11	1011	B
12	1100	C
13	1101	D
14	1110	E
15	1111	F

The number sixteen is represented as 10 in hexadecimal, because sixteen is **1** sixteen and **0** ones. Note $16_{10} = 10_{16} = 10000_2$.

Just as powers of two are fundamental to interpreting binary numbers, powers of sixteen are necessary for understanding hexadecimal. Here are the first few powers. (Memorizing these is *not* essential.)

Table 1.4 Powers of 16

$\cdots$	16^5	16^4	16^3	16^2	16^1	16^0
$\cdots$	$1,048,576$	$65,536$	4096	256	16	1

We can convert between hexadecimal and decimal in the same way that we converted between binary and decimal.

Example 1.3. Convert the hexadecimal number 1A2C to decimal.

Solution Simply write 1A2C as a sum of powers of sixteen in hexadecimal, then convert the sums to decimal. (In interpreting the first line, recall that 10 is the hexadecimal representation of sixteen, i.e., $10_{16} = 16_{10}$.)

$$\begin{aligned}
1\text{A}2\text{C} &= \mathbf{1}\cdot 10^3 + \mathbf{A}\cdot 10^2 + \mathbf{2}\cdot 10^1 + \mathbf{C}\cdot 10^0 && \text{(hexadecimal)}\\
&= \mathbf{1}\cdot 16^3 + \mathbf{10}\cdot 16^2 + \mathbf{2}\cdot 16^1 + \mathbf{12}\cdot 16^0 && \text{(decimal)}\\
&= \mathbf{1}\cdot 4096 + \mathbf{10}\cdot 256 + \mathbf{2}\cdot 16 + \mathbf{12}\cdot 1 && \text{(decimal)}\\
&= 6700 && \text{(decimal)}
\end{aligned}$$

Thus $1\text{A}2\text{C}_{16} = 6700_{10}$, that is, 1A2C (base 16) is 6700 (base 10). ✍

Converting between hexadecimal and binary is extremely simple. We illustrate the technique first, before explaining why it works. Suppose we wish to convert the binary number 111111001000001011 to hexadecimal. The first step is to divide the digits of this binary number into groups of four, beginning from the right.

$$11\ 1111\ 0010\ 0000\ 1011$$

If necessary, add extra zeros to left end of the left-most grouping, so that it too contains four digits.

$$0011\ 1111\ 0010\ 0000\ 1011$$

Now use Table 1.3 (or innate numerical reasoning) to convert each 4-digit binary grouping to the corresponding hexadecimal digit.

$$\begin{array}{ccccc} 0011 & 1111 & 0010 & 0000 & 1011 \\ 3 & \mathrm{F} & 2 & 0 & \mathrm{B} \end{array}$$

We conclude that $111111001000001011_2 = 3\mathrm{F}20\mathrm{B}_{16}$.

The reverse process works for converting hexadecimal to binary. Suppose we wanted to convert 1A2C to binary. Taking the reverse of the above approach (and using Table 1.3 if necessary), we write

$$\begin{array}{cccc} 1 & \mathrm{A} & 2 & \mathrm{C} \\ 0001 & 1010 & 0010 & 1100. \end{array}$$

Ignoring the three 0's on the far left, we see $1\mathrm{A}2\mathrm{C}_{16} = 1\ 1010\ 0010\ 1100_2$.

It is easy to see why this technique works. Just use the computation from Exercise 1.3 on page 11, but convert 1A2C to binary instead of decimal. (Here we use the fact that $10_{16} = 10000_2$.)

$$\begin{aligned} 1\mathrm{A}2\mathrm{C} &= \mathbf{1}\cdot 10^3 \ + \ \mathbf{A}\cdot 10^2 \ + \ \mathbf{2}\cdot 10^1 \ + \ \mathbf{C}\cdot 10^0 && \text{(base-16)} \\ &= \mathbf{1}\cdot 10000^3 + \mathbf{1010}\cdot 10000^2 + \mathbf{10}\cdot 10000^1 + \mathbf{1100}\cdot 10000^0 && \text{(binary).} \end{aligned}$$

Doing the addition in columns, we get:

$$\begin{array}{rrrrr} & \mathbf{1} & 0000 & 0000 & 0000 \\ & & \mathbf{1010} & 0000 & 0000 \\ & & & \mathbf{10} & 0000 \\ + & & & & \mathbf{1100} \\ \hline & 1 & 1010 & 0010 & 1100 \end{array}$$

This is the same number we would get by replacing each digit in 1A2C with its binary equivalent.

Exercises for Chapter 1

A. Convert the decimal number to binary and hexadecimal.

1. 347	**2.** 10,000	**3.** 2039	**4.** 64
5. 256	**6.** 257	**7.** 258	**8.** 258

B. Convert the binary number to hexadecimal and decimal.

9. 110110011	**10.** 10101010	**11.** 1111111	**12.** 111000111
13. 101101001	**14.** 10011010	**15.** 1000001	**16.** 100100101

C. Convert the hexadecimal number to decimal and binary

17. 123	**18.** ABC	**19.** 5A4D	**20.** F12
21. B0CA	**22.** C0FFEE	**23.** BEEF	**24.** ABBA

Solutions for Chapter 1

1. $347 = 256+64+16+8+2+1 = 1{\cdot}2^8+0{\cdot}2^7+1{\cdot}2^6+0{\cdot}2^5+1{\cdot}2^4+1{\cdot}2^3+1{\cdot}2^2+1{\cdot}2^1+1{\cdot}2^0$.
Thus $347_{10} = 101011011_2 = 0001\ 0101\ 1011_2 = 15\text{B}_{16}$.

3. $2039 = 1024 + 512 + 256 + 128 + 64 + 32 + 16 + 4 + 2 + 1 =$
$1 \cdot 2^{10} + 1 \cdot 2^9 + 1 \cdot 2^8 + 1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$.
Thus $2039_{10} = 11111110111_2 = 0111\ 1111\ 0111_2 = 7\text{F}7_{16}$.

5. $256 = 2^8 = \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$.
Thus $256_{10} = 100000000_2 = 0001\ 0000\ 0000_2 = 100_{16}$.

7. $258 = 256 + 2 = \cdot 2^8 + 0 \cdot 2^7 + 0 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$.
Thus $258_{10} = 100000010_2 = 0001\ 0000\ 0010_2 = 102_{16}$.

9. $110110011_2 = 0001\ 1011\ 0011_2 = 1\text{B}3_{16} = 1 \cdot 16^2 + 11 \cdot 16^1 + 3 \cdot 16^0 = 435_{10}$

11. $1111111_2 = 0111\ 1111_2 = 7F_{16} = 7 \cdot 16 + 15 = 127_{10}$.

13. $101101001_2 = 0001\ 0110\ 1001_2 = 169_{16} = 1 \cdot 16^2 + 6 \cdot 16 + 9 = 361_{10}$.

15. $1000001_2 = 0100\ 0001_2 = 41_{16} = 4 \cdot 16 + 1 = 65_{10}$.

17. $123_{16} = 1 \cdot 16^2 + 2 \cdot 16^1 + 3 \cdot 16^0 = 256 + 32 + 3 = 291_{10}$.
$123_{16} = 0001\ 0010\ 0011_2 = 100100011_2$.

19. $5\text{A}4\text{D}_{16} = 5{\cdot}16^3+10{\cdot}16^2+4{\cdot}16^1+13{\cdot}16^0 = 5{\cdot}4096+10{\cdot}256+4{\cdot}16+13 = 23117_{10}$.
$5\text{A}4\text{D}_{16} = 0101\ 1010\ 0100\ 1101_2 = 101101001001101_2$.

21. $\text{B0CA}_{16} = 11{\cdot}16^3+0{\cdot}16^2+12{\cdot}16^1+10{\cdot}16^0 = 11{\cdot}4096+0{\cdot}256+12{\cdot}16+10 = 45258_{10}$.
$\text{B0CA}_{16} = 1011\ 0000\ 1100\ 1010_2 = 101100001100_2$

23. $\text{BEEF}_{16} = 11{\cdot}16^3+14{\cdot}16^2+14{\cdot}16^1+15{\cdot}16^0 11{\cdot}4096+14{\cdot}256+14{\cdot}16+15 = 48879_{10}$.
$\text{BEEF}_{16} = 1011\ 1110\ 1110\ 1010_2 = 1011111011101010_2$.