Chapter 18

Functions

You know from calculus that functions play a fundamental role in mathematics. You likely view a function as a kind of formula that describes a relationship between two (or more) quantities. You certainly understand and appreciate the fact that relationships between quantities are important in all scientific disciplines, so you do not need to be convinced that functions are important. Still, you may not be aware of the full significance of functions. Functions are more than merely descriptions of numeric relationships. In a more general sense, functions can compare and relate different kinds of mathematical structures. You will see this as your understanding of mathematics deepens. In preparation of this deepening, we will now explore a more general and versatile view of functions.

The concept of a relation between sets (Definition 17.7) plays a big role here, so you may want to quickly review it.

18.1 Functions

Let's start on familiar ground. Consider the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$. Its graph is the set of points $R = \big\{(x, x^2) : x \in \mathbb{R}\big\} \subseteq \mathbb{R} \times \mathbb{R}$.

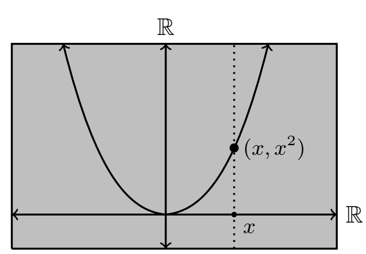


Fig. 18.1 A familiar function

Having read Chapter 17, you may see f in a new light. Its graph $R \subseteq \mathbb{R} \times \mathbb{R}$ is a relation on the set $\mathbb{R}$. In fact, as we shall see, functions are just special kinds of relations. Before stating the exact definition, we look at another example. Consider the function $f(n) = |n| + 2$ that converts integers n into natural numbers $|n| + 2$. Its graph is $R = \{(n, |n| + 2) : n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{N}$.

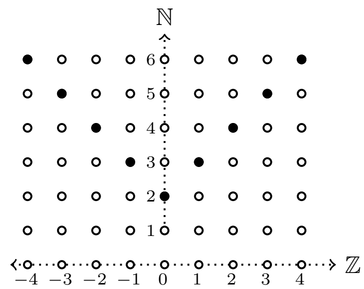


Fig. 18.2 The function $f : \mathbb{Z} \to \mathbb{N}$, where $f(n) = |n| + 2$

Figure 18.2 shows the graph R as darkened dots in the grid of points $\mathbb{Z} \times \mathbb{N}$. Notice that here R is not a relation on a single set. The set of input values $\mathbb{Z}$ is different from the set $\mathbb{N}$ of output values, so the graph $R \subseteq \mathbb{Z} \times \mathbb{N}$ is *a relation from* $\mathbb{Z}$ *to* $\mathbb{N}$.

This example illustrates three things. First, a function can be viewed as sending elements from one set A to another set B. (In the case of f, $A = \mathbb{Z}$ and $B = \mathbb{N}$.) Second, such a function can be regarded as a relation from A to B. Third, for every input value n, there is *exactly one* output value $f(n)$. In your high school algebra course, this was expressed by the *vertical line test*: Any vertical line intersects a function's graph at most once. It means that for any input value x, the graph contains exactly one point of form $(x, f(x))$. Our main definition, given below, incorporates all of these ideas.

Definition 18.1. Suppose A and B are sets. A **function** f from A to B (denoted as $f : A \to B$) is a relation $f \subseteq A \times B$ from A to B, satisfying the property that for each $a \in A$ the relation f contains exactly one ordered pair of form (a, b). The statement $(a, b) \in f$ is abbreviated $f(a) = b$.

Example 18.1. Consider the function f from Figure 18.2. According to Definition‘18.1, we view f as the set of points in its graph, that is, $f = \{(n, |n|+2) : n \in \mathbb{Z}\} \subseteq \mathbb{Z} \times \mathbb{N}$. This is a relation from $\mathbb{Z}$ to $\mathbb{N}$, and indeed given any $a \in \mathbb{Z}$ the set f contains exactly one ordered pair $(a, |a| + 2)$ whose first coordinate is a. Since $(1, 3) \in f$, we write $f(1) = 3$; and since $(-3, 5) \in f$ we write $f(-3) = 5$, etc. In general, $(a, b) \in f$ means that f sends the input value a to the output value b, and we express this as $f(a) = b$. This function can be expressed by a formula: For each input value n, the output value is $|n| + 2$, so we may write $f(n) = |n| + 2$. All this agrees with the way we thought of functions in algebra and calculus; the only difference is that now we also think of a function as a relation. ✍

Definition 18.2. For a function $f\colon A \to B$, the set A is called the **domain** of f. (Think of the domain as the set of possible "input values" for f.) The set B is called the **codomain** of f. The **range** of f is the set $\big\{f(a) : a \in A\big\} = \big\{b : (a,b) \in f\big\}$. (Think of the range as the set of all possible "output values" for f. Think of the codomain as a sort of "target" for the outputs.)

Continuing Example 18.1, the domain of f is $\mathbb{Z}$ and its codomain is $\mathbb{N}$. Its range is $\big\{f(a) : a \in \mathbb{Z}\big\} = \big\{|a|+2 : a \in \mathbb{Z}\big\} = \big\{2,3,4,5,\dots\big\}$. The range is a subset of the codomain, but it does not (in this case) equal the codomain.

In our examples so far, the domains and codomains are sets of numbers, but this needn't be the case in general, as the next example indicates.

Example 18.2. Let $A = \big\{p,q,r,s\big\}$ and $B = \big\{0,1,2\big\}$, and

$$f = \big\{(p,0),(q,1),(r,2),(s,2)\big\} \subseteq A \times B.$$

This is a function $f\colon A \to B$ because each element of A occurs exactly once as a first coordinate of an ordered pair in f. Note $f(p) = 0$, $f(q) = 1$, $f(r) = 2$ and $f(s) = 2$. The domain of f is $\big\{p,q,r,s\big\}$. The codomain and range are both $\big\{0,1,2\big\}$. ✍

If A and B are not both sets of numbers it can be difficult to draw a graph of $f\colon A \to B$ in the traditional sense. Figure 18.3(a) shows an attempt at a graph of f from Example 18.2. The sets A and B are aligned roughly as x- and y-axes, and the Cartesian product $A \times B$ is filled in accordingly. The subset $f \subseteq A \times B$ is indicated with dashed lines, and this can be regarded as a "graph" of f. Figure 18.3(b) shows a more natural depiction of f. Sets A and B are drawn side-by-side, and arrows point from a to b when $f(a) = b$.

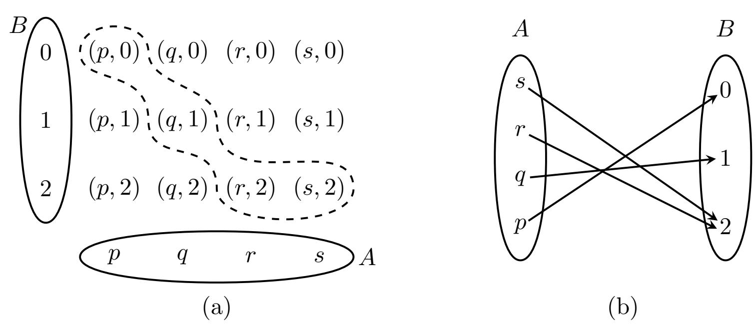


Fig. 18.3 Two ways of drawing the function $f = \big\{(p,0),(q,1),(r,2),(s,2)\big\}$

In general, if $f\colon A \to B$ is the kind of function you may have encountered in algebra or calculus, then conventional graphing methods offer the best visual description of it. But if A and B are finite or if we are thinking of them as generic sets, then describing f with arrows is often a more appropriate visual representation.

We emphasize that, according to Definition 18.1, a function is just a special kind of set: a function $f\colon A \to B$ is a subset of $A \times B$. By contrast, your calculus text probably defined a function as a certain kind of "rule." While that intuitive outlook is adequate for the first few semesters of calculus, it does not hold up well to the rigorous mathematical standards necessary for further progress. The problem is that the word "rule" is too vague. Definition 18.1 removes the ambiguity and makes a function a concrete mathematical object. It allows us, for example, to talk about about a "set of functions" and know exactly what we are speaking of. A set of functions is just a of set of sets. Such precision is necessary in proofs.

Still, in practice we tend to think of functions as rules. Given $f\colon \mathbb{Z} \to \mathbb{N}$ where $f(x) = |x| + 2$, we think of this as a rule that associates any number $n \in \mathbb{Z}$ to the number $|n|+2$ in $\mathbb{N}$, rather than a set containing ordered pairs $(n, |n|+2)$. It is only when we have to understand or interpret the theoretical nature of functions (as we do in this text) that Definition 18.1 comes to bear. The definition is a foundation that gives us license to think about functions in a more informal way.

The next example brings up a point about notation. Consider a function such as $f\colon \mathbb{Z}^2 \to \mathbb{Z}$, whose domain is a Cartesian product. This function takes as input an ordered pair $(m,n) \in \mathbb{Z}^2$ and sends it to a number $f((m,n)) \in \mathbb{Z}$. For brevity, we usually write $f(m,n)$ instead of $f((m,n))$, even though this is like writing fx instead of $f(x)$. Also, although we've been using the letters f, g and h for functions, any other reasonable symbol could be used. Greek letters such as φ and θ are common.

Example 18.3. Say a function $\varphi : \mathbb{Z}^2 \to \mathbb{Z}$ is defined as $\varphi(m,n) = 6m - 9n$. Note that as a set, this function is $\varphi = \big\{ \big((m,n), 6m-9n\big) : (m,n) \in \mathbb{Z}^2 \big\} \subseteq \mathbb{Z}^2 \times \mathbb{Z}$. What is the range of φ?

Solution: First note that for any $(m,n) \in \mathbb{Z}^2$, the value $f(m,n) = 6m - 9n = 3(2m-3n)$ is a multiple of 3. Thus every number in the range is a multiple of 3, so the range is a *subset* of the set of all multiples of 3. On the other hand, if $b = 3k$ is a multiple of 3, then $\varphi(-k,-k) = 6(-k) - 9(-k) = 3k = b$, so any multiple of 3 is in the range. So the range of φ is the set $\big\{3k : k \in \mathbb{Z}\big\}$ of all multiples of 3. ✍

To conclude this section, let's use Definition 18.1 to help us understand what it means for two functions $f\colon A \to B$ and $g\colon C \to D$ to be equal. According to our definition, functions f and g are subsets $f \subseteq A \times B$ and $g \subseteq C \times D$. It makes sense to say that f and g are equal if $f = g$, that is, if they are equal as sets.

Thus the two functions $f = \big\{(1,a),(2,a),(3,b)\big\}$ and $g = \big\{(3,b),(2,a),(1,a)\big\}$ are equal because the sets f and g are equal. Notice that the domain of both functions is $A = \big\{1,2,3\big\}$, the set of first elements x in the ordered pairs $(x,y) \in f = g$. In general, equal functions must have equal domains.

Note that $f = g$ means $f(x) = g(x)$ for each $x \in A$. This yields our definition.

Definition 18.3. Two functions $f\colon A \to B$ and $g\colon A \to D$ are said to be **equal** if $f(x) = g(x)$ for every $x \in A$.

Observe that f and g can have different codomains and still be equal. Consider the functions $f\colon \mathbb{Z} \to \mathbb{N}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = |x| + 2$ and $g(x) = |x| + 2$. Even though their codomains are different, the functions are equal because $f(x) = g(x)$ for every x in the domain.

Exercises for Section 18.1

1. Let $A = \{0,1,2,3,4\}$, $B = \{2,3,4,5\}$ and $f = \{(0,3),(1,3),(2,4),(3,2),(4,2)\}$. State the domain and range of f. Find $f(2)$ and $f(1)$.

2. Let $A = \{a,b,c,d\}$, $B = \{2,3,4,5,6\}$ and $f = \{(a,2),(b,3),(c,4),(d,5)\}$. State the domain and range of f. Find $f(b)$ and $f(d)$.

3. There are four different functions $f\colon \{a,b\} \to \{0,1\}$. List them all. Diagrams will suffice.

4. There are eight different functions $f\colon \{a,b,c\} \to \{0,1\}$. List them all. Diagrams will suffice.

5. Give an example of a relation from $\{a,b,c,d\}$ to $\{d,e\}$ that is not a function.

6. Suppose $f\colon \mathbb{Z} \to \mathbb{Z}$ is defined as $f = \{(x, 4x+5) : x \in \mathbb{Z}\}$. State the domain, codomain and range of f. Find $f(10)$.

7. Consider the set $f = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : 3x + y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

8. Consider the set $f = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x + 3y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$? Explain.

9. Consider the set $f = \{(x^2, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

10. Consider the set $f = \{(x^3, x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

11. Is the set $\theta = \{(X, |X|) : X \subseteq \mathbb{Z}_5\}$ a function? If so, what is its domain and range?

12. Is the set $\theta = \{((x,y),(3y, 2x, x+y)) : x, y \in \mathbb{R}\}$ a function? If so, what is its domain, codomain and range?

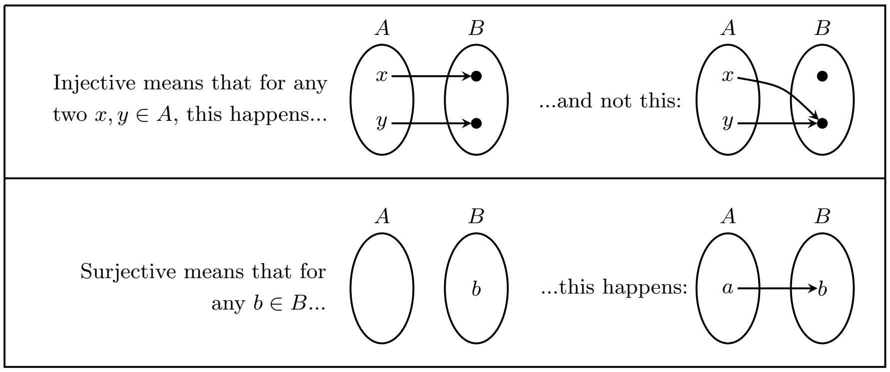
18.2 Injective and Surjective Functions

You may recall from algebra and calculus that a function may be *one-to-one* and *onto*, and these properties are related to whether or not the function is invertible. We now review these important ideas. In advanced mathematics, the word *injective* is often used instead of *one-to-one*, and *surjective* is used instead of *onto*. Here are the exact definitions:

Definition 18.4. A function $f\colon A \to B$ is:

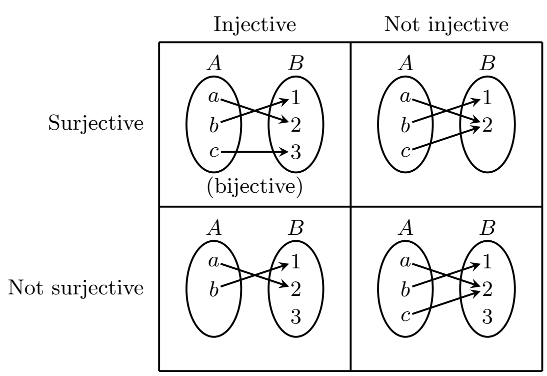
- **injective** (or one-to-one) if for every $x, y \in A$, $x \neq y$ implies $f(x) \neq f(y)$;
- **surjective** (or onto) if for every $b \in B$ there is an $a \in A$ with $f(a) = b$;
- **bijective** if f is both injective and surjective.

Below is a visual description of Definition 18.4. In essence, injective means that unequal elements in A always get sent to unequal elements in B. Surjective means that every element of B has an arrow pointing to it, that is, it equals $f(a)$ for some a in the domain of f.



For more concrete examples, consider the following functions from $\mathbb{R}$ to $\mathbb{R}$. The function $f(x) = x^2$ is not injective because $-2 \neq 2$, but $f(-2) = f(2)$. Nor is it surjective, for if $b = -1$ (or if b is any negative number), then there is no $a \in \mathbb{R}$ with $f(a) = b$. On the other hand, $g(x) = x^3$ is both injective and surjective, so it is also bijective.

There are four possible injective/surjective combinations that a function may possess. This is illustrated in the following figure showing four functions from A to B. Functions in the first column are injective, those in the second column are not injective. Functions in the first row are surjective, those in the second row are not.



We note in passing that, according to the definitions, a function is surjective if and only if its codomain equals its range.

Often it is necessary to prove that a particular function $f\colon A \to B$ is injective. For this we must prove that for any two elements $x, y \in A$, the conditional statement $(x \neq y) \Rightarrow \big(f(x) \neq f(y)\big)$ is true. The two main approaches for this are summarized below.

How to show a function $f\colon A \to B$ is injective:

Direct approach:	**Contrapositive approach:**
Suppose $x, y \in A$ and $x \neq y$.	Suppose $x, y \in A$ and $f(x) = f(y)$.
$\vdots$	$\vdots$
Therefore $f(x) \neq f(y)$.	Therefore $x = y$.

Of these two approaches, the contrapositive is often the easiest to use, especially if f is defined by an algebraic formula. This is because the contrapositive approach starts with the *equation* $f(x) = f(y)$ and proceeds to the *equation* $x = y$. In algebra, as you know, it is usually easier to work with equations than inequalities.

To prove that a function is *not* injective, you must *disprove* the statement $(x \neq y) \Rightarrow \big(f(x) \neq f(y)\big)$. For this it suffices to find example of two elements $x, y \in A$ for which $x \neq y$ and $f(x) = f(y)$.

Next we examine how to prove that $f\colon A \to B$ is *surjective*. According to Definition 18.4, we must prove the statement $\forall\, b \in B, \exists\, a \in A, f(a) = b$. In words, we must show that for any $b \in B$, there is at least one $a \in A$ (which may depend on b) having the property that $f(a) = b$. Here is an outline.

How to show a function $f\colon A \to B$ is surjective:

Suppose $b \in B$.
[Prove there exists $a \in A$ for which $f(a) = b$.]

In the second step, we have to prove the existence of an a for which $f(a) = b$. For this, just finding an example of such an a would suffice. (How to find such an example depends on how f is defined. If f is given as a formula, we may be able to find a by solving the equation $f(a) = b$ for a. Sometimes you can find a by just plain common sense.) To show f is *not* surjective, we must prove the negation of $\forall\, b \in B, \exists\, a \in A, f(a) = b$, that is, we must prove $\exists\, b \in B, \forall\, a \in A, f(a) \neq b$.

The following examples illustrate these ideas.

Example 18.4. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^3 + 1$ is injective and surjective.

Solution: To show that it is injective, we take the contrapositive approach. Let $x, y \in \mathbb{R}$ and assume $f(x) = f(y)$. Then $x^3 + 1 = y^3 + 1$, which reduces to $x^3 = y^3$. Taking the cube root of both sides yields $x = y$. Therefore f is injective.

To show that it is surjective, take any b in the codomain $\mathbb{R}$. Let $a = \sqrt[3]{b-1}$. Then $f(a) = f\left(\sqrt[3]{b-1}\right) = \left(\sqrt[3]{b-1}\right)^3 + 1 = b$. This shows that f is surjective. ✍

The next example involves the set $\mathbb{R} - \{0\}$, which is $\mathbb{R}$ with the number 0 removed.

Example 18.5. Show that the function $f\colon \mathbb{R} - \{0\} \to \mathbb{R}$ defined as $f(x) = \dfrac{1}{x} + 1$ is injective but not surjective.

Solution: We will use the contrapositive approach to show that f is injective. To this end, suppose $x, y \in \mathbb{R} - \{0\}$ and $f(x) = f(y)$. This means $\dfrac{1}{x} + 1 = \dfrac{1}{y} + 1$. Subtracting 1 from both sides and inverting produces $x = y$. Therefore f is injective.

The function f is not surjective because there exists an element $b = 1 \in \mathbb{R}$ for which $f(x) = \dfrac{1}{x} + 1 \neq 1$ for every $x \in \mathbb{R} - \{0\}$. ✍

Example 18.6. Show that the function $g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by the formula $g(m, n) = (m + n, m + 2n)$, is both injective and surjective.

Solution: We will use the contrapositive approach to show that g is injective. Thus we need to show that $g(m, n) = g(k, \ell)$ implies $(m, n) = (k, \ell)$. Suppose $(m, n), (k, \ell) \in \mathbb{Z} \times \mathbb{Z}$ and $g(m, n) = g(k, \ell)$. Then $(m + n, m + 2n) = (k + \ell, k + 2\ell)$. It follows that $m + n = k + \ell$ and $m + 2n = k + 2\ell$. Subtracting the first equation from the second gives $n = \ell$. Next, subtract $n = \ell$ from $m + n = k + \ell$ to get $m = k$. Since $m = k$ and $n = \ell$, it follows that $(m, n) = (k, \ell)$. Therefore g is injective.

To see that g is surjective, consider an arbitrary element $(b, c) \in \mathbb{Z} \times \mathbb{Z}$. We need to show that there is some $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ for which $g(x, y) = (b, c)$. To find (x, y), note that $g(x, y) = (b, c)$ means $(x + y, x + 2y) = (b, c)$. This leads to the following system of equations:

$$\begin{aligned} x + y &= b \\ x + 2y &= c. \end{aligned}$$

Solving gives $x = 2b - c$ and $y = c - b$. Then $(x, y) = (2b - c, c - b)$. We now have $g(2b - c, c - b) = (b, c)$, and it follows that g is surjective. ✍

Example 18.7. Consider function $h\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ defined as $h(m, n) = \dfrac{m}{|n| + 1}$. Determine whether this is injective and whether it is surjective.

Solution: This function is *not* injective because of the unequal elements $(1, 2)$ and $(1, -2)$ in $\mathbb{Z} \times \mathbb{Z}$ for which $h(1, 2) = h(1, -2) = \dfrac{1}{3}$.

However, h is surjective, as follows Take any element $b \in \mathbb{Q}$. Then $b = \dfrac{c}{d}$ for some $c, d \in \mathbb{Z}$. Notice we may assume d is positive by making c negative, if necessary. Then $h(c, d - 1) = \dfrac{c}{|d - 1| + 1} = \dfrac{c}{d} = b$. ✍

Exercises for Section 18.2

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Give an example of a function $f\colon A \to B$ that is neither injective nor surjective.

2. Consider the logarithm function $\ln : (0, \infty) \to \mathbb{R}$. Decide whether this function is injective and whether it is surjective.

3. Consider the cosine function $\cos : \mathbb{R} \to \mathbb{R}$. Decide whether this function is injective and whether it is surjective. What if it had been defined as $\cos : \mathbb{R} \to [-1, 1]$?

4. A function $f\colon \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ is defined as $f(n) = (2n, n+3)$. Verify whether this function is injective and whether it is surjective.

5. A function $f\colon \mathbb{Z} \to \mathbb{Z}$ is defined as $f(n) = 2n + 1$. Verify whether this function is injective and whether it is surjective.

6. A function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m, n) = 3n - 4m$. Verify whether this function is injective and whether it is surjective.

7. A function $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is defined as $f(m, n) = 2n - 4m$. Verify whether this function is injective and whether it is surjective.

8. A function $f\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ is defined as $f(m, n) = (m+n, 2m+n)$. Verify whether this function is injective and whether it is surjective.

9. Prove that the function $f\colon \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \dfrac{5x+1}{x-2}$ is bijective.

10. Prove the function $f\colon \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \left(\dfrac{x+1}{x-1}\right)^3$ is bijective.

11. Consider the function $\theta : \{0, 1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a, b) = (-1)^a b$. Is θ injective? Is it surjective? Bijective? Explain.

12. Consider the function $\theta : \{0, 1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a, b) = a - 2ab + b$. Is θ injective? Is it surjective? Bijective? Explain.

13. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = (xy, x^3)$. Is f injective? Is it surjective? Bijective? Explain.

14. Consider the function $\theta : \mathscr{P}(\mathbb{Z}) \to \mathscr{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$. Is θ injective? Is it surjective? Bijective? Explain.

15. This question concerns functions $f\colon \{A, B, C, D, E, F, G\} \to \{1, 2, 3, 4, 5, 6, 7\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

16. This question concerns functions $f\colon \{A, B, C, D, E\} \to \{1, 2, 3, 4, 5, 6, 7\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

17. This question concerns functions $f\colon \{A, B, C, D, E, F, G\} \to \{1, 2\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

18. Prove that the function $f\colon \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = \dfrac{(-1)^n(2n-1)+1}{4}$ is bijective.

18.3 The Pigeonhole Principle Revisited

We first encountered a result called the *pigeonhole principle* in Section 6.9. It turns out that the pigeonhole principle has a useful phrasing in the language of injective and surjective functions, and we now discuss this. Our discussion will not use any material from Chapter 6, so it does not matter if you skipped it.

The pigeonhole principle is motivated by a simple thought experiment: Imagine there is a set A of pigeons and a set B of pigeonholes, and all the pigeons fly into the pigeonholes. You can think of this as describing a function $f\colon A \to B$, where pigeon p flies into pigeonhole $f(p)$. See Figure 18.4.

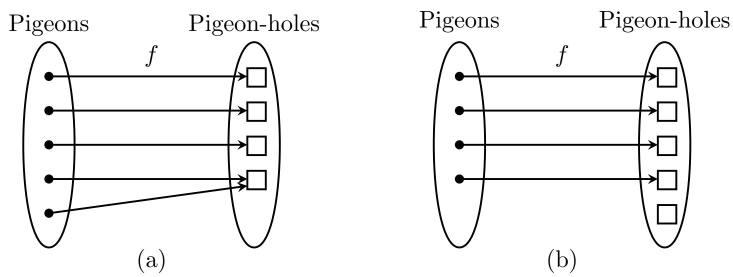


Fig. 18.4 The pigeonhole principle

In Figure 18.4(a) there are more pigeons than pigeonholes, and it is obvious that in such a case at least two pigeons have to fly into the same pigeonhole, meaning that f is not injective. In Figure 18.4(b) there are fewer pigeons than pigeonholes, so clearly at least one pigeonhole remains empty, meaning that f is not surjective.

This simple idea is called the *pigeonhole principle*. We encountered it first in Section 6.9, but we restate it here in the language of functions.

Fact 18.1. The Pigeonhole Principle (function version)
Suppose A and B are finite sets and $f\colon A \to B$ is any function. Then:

- If $|A| > |B|$, then f is not injective.
- If $|A| < |B|$, then f is not surjective.

Though the pigeonhole principle is obvious, it can be used to prove some things that are not so obvious. Two examples follow

Example 18.8. Prove the following proposition.

Proposition. 0.5em If A is any set of 10 integers between 1 and 100, then there exist two different subsets $X \subseteq A$ and $Y \subseteq A$ for which the sum of elements in X equals the sum of elements in Y.

To illustrate what this proposition is saying, consider the random set

$$A = \{5, 7, 12, 11, 17, 50, 51, 80, 90, 100\}$$

of 10 integers between 1 and 100. Notice that A has subsets $X = \{5, 80\}$ and $Y = \{7, 11, 17, 50\}$ for which the sum of the elements in X equals the sum of those in Y. If we tried to "mess up" A by changing the 5 to a 6, we get

$$A = \{6, 7, 12, 11, 17, 50, 51, 80, 90, 100\}$$

which has subsets $X = \{7, 12, 17, 50\}$ and $Y = \{6, 80\}$ both of whose elements add up to the same number (86). The proposition asserts that this is always possible, no matter what A is. Here is a proof.

Proof. Suppose $A \subseteq \{1, 2, 3, 4, \ldots, 99, 100\}$ and $|A| = 10$, as stated. Notice that if $X \subseteq A$, then X has no more than 10 elements, each between 1 and 100, and therefore the sum of all the elements of X is less than $100 \cdot 10 = 1000$. Consider the function

$$f : \mathscr{P}(A) \to \{0, 1, 2, 3, 4, \ldots, 1000\}$$

where $f(X)$ is the sum of the elements in X. (Examples: $f(\{3, 7, 50\}) = 60$; $f(\{1, 70, 80, 95\}) = 246$.) As $|\mathscr{P}(A)| = 2^{10} = 1024 > 1001 = |\{0, 1, 2, 3, \ldots, 1000\}|$, it follows from the pigeonhole principle that f is not injective. Therefore there are two unequal sets $X, Y \in \mathscr{P}(A)$ for which $f(X) = f(Y)$. In other words, there are subsets $X \subseteq A$ and $Y \subseteq A$ for which the sum of elements in X equals the sum of elements in Y. □

Example 18.9. Prove the following proposition.

Proposition. There are two Texans with the same number of hairs on their heads.

Proof. We will use two facts. First, the population of Texas is more than twenty million. Second, it is a biological fact that every human head has fewer than one million hairs. Let A be the set of all Texans, and let $B = \{0, 1, 2, 3, 4, \ldots, 1000000\}$. Let $f : A \to B$ be the function for which $f(x)$ equals the number of hairs on the head of x. Since $|A| > |B|$, the pigeonhole principle asserts that f is not injective. Thus there are two Texans x and y for whom $f(x) = f(y)$, meaning that they have the same number of hairs on their heads. □

Proofs that use the pigeonhole principle tend to be inherently non-constructive, in the sense discussed in Section 13.4. For example, the above proof does not explicitly give us of two Texans with the same number of hairs on their heads; it only shows that two such people exist. If we were to make a constructive proof, we could find examples of two bald Texans. Then they have the same number of head hairs, namely zero.

Exercises for Section 18.3

1. Prove that if six integers are chosen at random, then at least two of them will have the same remainder when divided by 5.

2. Prove that if a is a natural number, then there exist two unequal natural numbers k and ℓ for which $a^k - a^\ell$ is divisible by 10.

3. Prove that for any six integers, 9 divides the sum or difference of two of them.

4. Consider a square whose side-length is one unit. Select any five points from inside this square. Prove that at least two of these points are within $\frac{\sqrt{2}}{2}$ units of each other.

5. Prove that any set of seven distinct integers contains a pair of integers whose sum or difference is divisible by 10.

6. Given a sphere S, a *great circle* of S is the intersection of S with a plane through its center. Every great circle divides S into two parts. A *hemisphere* is the union of the great circle and one of these two parts. Prove that if five points are placed arbitrarily on S, then there is a hemisphere that contains four of them.

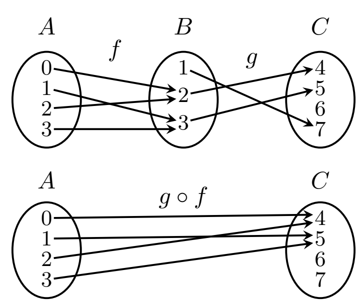
7. Prove or disprove: Any subset $X \subseteq \{1, 2, 3, \dots, 2n\}$ with $|X| > n$ contains two (unequal) elements for which one divides the other.

18.4 Composition

You should be familiar with function composition from algebra and calculus. Still, it is worthwhile to revisit it now with our more sophisticated ideas about functions.

Definition 18.5. Suppose $f\colon A \to B$ and $g\colon B \to C$ are functions for which the codomain of f is the domain of g. The **composition** of f with g is another function, denoted as $g\circ f$ and defined as follows: If $x \in A$, then $g\circ f\,(x) = g(f(x))$. Therefore $g\circ f$ sends elements of A to elements of C, so $g\circ f\colon A \to C$.

The diagram below illustrates this. Here $f\colon A \to B$, $g\colon B \to C$, and $g\circ f\colon A \to C$. We have, for example, $g\circ f(0) = g(f(0)) = g(2) = 4$. Be careful with the order of the symbols. Even though g comes first in the symbol $g\circ f$, we work out $g\circ f(x)$ as $g(f(x))$, with f acting on x first, followed by g acting on $f(x)$.



Notice that the composition $g \circ f$ also makes sense if the range of f is a *subset* of the domain of g. You should take note of this fact, but to keep things simple we will emphasize situations where the codomain of f equals the domain of g.

Example 18.10. Suppose $A = \{a, b, c\}$, $B = \{0, 1\}$, $C = \{1, 2, 3\}$. Let $f: A \to B$ be the function $f = \{(a, 0), (b, 1), (c, 0)\}$, and let $g: B \to C$ be the function $g = \{(0, 3), (1, 1)\}$. Then $g \circ f = \{(a, 3), (b, 1), (c, 3)\}$. ✍

Example 18.11. Say $A = \{a, b, c\}$, $B = \{0, 1\}$, $C = \{1, 2, 3\}$. Let $f: A \to B$ be the function $f = \{(a, 0), (b, 1), (c, 0)\}$. Let $g: C \to B$ be $g = \{(1, 0), (2, 1), (3, 1)\}$. Here the composition $g \circ f$ is not defined because the codomain B of f is not the same set as the domain C of g. Remember: In order for $g \circ f$ to make sense, the codomain of f must equal the domain of g. (Or at least be a subset of it.) ✍

Example 18.12. Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2 + x$, and $g: \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = x + 1$. Then $g \circ f: \mathbb{R} \to \mathbb{R}$ is the function defined by the formula $g \circ f\,(x) = g(f(x)) = g(x^2 + x) = x^2 + x + 1$.

Since the domains and codomains of g and f are the same, we can in this case do a composition in the other order. Note that $f \circ g: \mathbb{R} \to \mathbb{R}$ is the function defined as $f \circ g\,(x) = f(g(x)) = f(x + 1) = (x + 1)^2 + (x + 1) = x^2 + 3x + 2$.

This shows that even if $g \circ f$ and $f \circ g$ are both defined, they may not be equal. We can express this fact by saying *function composition is not commutative.* ✍

We close this section by proving several facts about function composition that you are likely to encounter in your future study of mathematics. First, we note that, although it is not commutative, function composition *is* associative.

Theorem 18.2. *Composition of functions is associative. That is if $f: A \to B$, $g: B \to C$ and $h: C \to D$, then $(h \circ g) \circ f = h \circ (g \circ f)$.*

Proof. Suppose f, g, h are as stated. Definition 18.5 implies that both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are functions from A to D. To show that they are equal, we just need to show

$$\Big((h \circ g) \circ f\Big)(x) = \Big(h \circ (g \circ f)\Big)(x)$$

for every $x \in A$. Note that Definition 18.5 yields

$$\Big((h \circ g) \circ f\Big)(x) = (h \circ g)(f(x)) = h(g(f(x))).$$

Also

$$\Big(h \circ (g \circ f)\Big)(x) = h(g \circ f\,(x)) = h(g(f(x))).$$

Thus $\Big((h \circ g) \circ f\Big)(x) = \Big(h \circ (g \circ f)\Big)(x)$, as both sides equal $h(g(f(x)))$. □

Theorem 18.3. *Suppose $f\colon A \to B$ and $g\colon B \to C$. If both f and g are injective, then $g \circ f$ is injective. If both f and g are surjective, then $g \circ f$ is surjective.*

Proof. First suppose both f and g are injective. To see that $g \circ f$ is injective, we must show that $g \circ f\,(x) = g \circ f\,(y)$ implies $x = y$. Suppose $g \circ f\,(x) = g \circ f\,(y)$. This means $g(f(x)) = g(f(y))$. It follows that $f(x) = f(y)$. (For otherwise g wouldn't be injective.) But since $f(x) = f(y)$ and f is injective, it must be that $x = y$. Therefore $g \circ f$ is injective.

Next suppose both f and g are surjective. To see that $g \circ f$ is surjective, we must show that for any element $c \in C$, there is a corresponding element $a \in A$ for which $g \circ f\,(a) = c$. Thus consider an arbitrary $c \in C$. Because g is surjective, there is an element $b \in B$ for which $g(b) = c$. And because f is surjective, there is an element $a \in A$ for which $f(a) = b$. Therefore $g(f(a)) = g(b) = c$, which means $g \circ f\,(a) = c$. Thus $g \circ f$ is surjective. □

Exercises for Section 18.4

1. Suppose $A = \{5, 6, 8\}$, $B = \{0, 1\}$, $C = \{1, 2, 3\}$. Let $f\colon A \to B$ be the function $f = \{(5, 1), (6, 0), (8, 1)\}$, and $g\colon B \to C$ be $g = \{(0, 1), (1, 1)\}$. Find $g \circ f$.

2. Suppose $A = \{1, 2, 3, 4\}$, $B = \{0, 1, 2\}$, $C = \{1, 2, 3\}$. Let $f\colon A \to B$ be

$$f = \{(1, 0), (2, 1), (3, 2), (4, 0)\},$$

and $g\colon B \to C$ be $g = \{(0, 1), (1, 1), (2, 3)\}$. Find $g \circ f$.

3. Suppose $A = \{1, 2, 3\}$. Let $f\colon A \to A$ be the function $f = \{(1, 2), (2, 2), (3, 1)\}$, and let $g\colon A \to A$ be the function $g = \{(1, 3), (2, 1), (3, 2)\}$. Find $g \circ f$ and $f \circ g$.

4. Suppose $A = \{a, b, c\}$. Let $f\colon A \to A$ be the function $f = \{(a, c), (b, c), (c, c)\}$, and let $g\colon A \to A$ be the function $g = \{(a, a), (b, b), (c, a)\}$. Find $g \circ f$ and $f \circ g$.

5. Consider the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sqrt[3]{x+1}$ and $g(x) = x^3$. Find the formulas for $g \circ f$ and $f \circ g$.

6. Consider the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x^2+1}$ and $g(x) = 3x + 2$. Find the formulas for $g \circ f$ and $f \circ g$.

7. Consider the functions $f, g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (mn, m^2)$ and $g(m, n) = (m + 1, m + n)$. Find the formulas for $g \circ f$ and $f \circ g$.

8. Consider the functions $f, g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (3m - 4n, 2m + n)$ and $g(m, n) = (5m + n, m)$. Find the formulas for $g \circ f$ and $f \circ g$.

9. Consider the functions $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f(m, n) = m + n$ and $g\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $g(m) = (m, m)$. Find the formulas for $g \circ f$ and $f \circ g$.

10. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = (xy, x^3)$. Find a formula for $f \circ f$.

18.5 Inverse Functions

You may recall from calculus that if a function f is injective and surjective, then it has an inverse function f^{-1} that "undoes" the effect of f in the sense that $f^{-1}(f(x)) = x$. (For example, if $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$.) We now review these ideas. Our approach uses two ingredients, outlined in the following definitions.

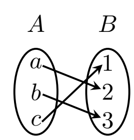
Definition 18.6. Given a set A, the **identity function** on A is the function $i_A : A \to A$ defined as $i_A(x) = x$ for every $x \in A$.

For instance, if $A = \{1, 2, 3\}$, then $i_A = \{(1,1), (2,2), (3,3)\}$. Also, the identity function on $\mathbb{Z}$ is $i_{\mathbb{Z}} = \{(n, n) : n \in \mathbb{Z}\}$. The identity function on a set is the function that sends any element of the set to itself.

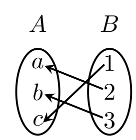
Notice that for any set A, the identity function i_A is bijective: It is injective because $i_A(x) = i_A(y)$ immediately reduces to $x = y$. It is surjective because if we take any element b in the codomain A, then b is also in the domain A, and $i_A(b) = b$.

Definition 18.7. Given a relation R from A to B, the **inverse relation of** R is the relation from B to A defined as $R^{-1} = \{(y, x) : (x, y) \in R\}$. In other words, the inverse of R is the relation R^{-1} obtained by interchanging the elements in every ordered pair in R.

For example, let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, and suppose f is the relation $f = \{(a, 2), (b, 3), (c, 1)\}$ from A to B. Then $f^{-1} = \{(2, a), (3, b), (1, c)\}$, and this is a relation from B to A. Notice that f is actually a function from A to B, and f^{-1} is a function from B to A. These two relations are drawn below. Notice the drawing for relation f^{-1} is just the drawing for f with arrows reversed.

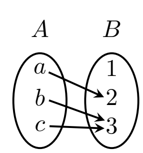


$f = \{(a, 2), (b, 3), (c, 1)\}$

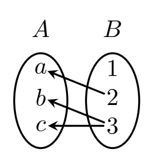


$f^{-1} = \{(2, a), (3, b), (1, c)\}$

For another example, let A and B be the same sets as above, but consider the relation $g = \{(a, 2), (b, 3), (c, 3)\}$ from A to B. Then $g^{-1} = \{(2, a), (3, b), (3, c)\}$ is a relation from B to A. These two relations are sketched below.



$g = \{(a, 2), (b, 3), (c, 3)\}$



$g^{-1} = \{(2, a), (3, b), (3, c)\}$

This time, even though the relation g is a function, its inverse g^{-1} is not a function (Because 3 occurs twice as a first coordinate of an ordered pair in g^{-1}.)

In the above examples, relations f and g are both functions, and f^{-1} is a function and g^{-1} is not. This raises a question: What properties does f have and g lack that makes f^{-1} a function and g^{-1} not a function? The answer is not hard to see. Function g is not injective because $g(b) = g(c) = 3$, and thus $(b, 3)$ and $(c, 3)$ are both in g. This causes a problem with g^{-1} because it means $(3, b)$ and $(3, c)$ are both in g^{-1}, so g^{-1} can't be a function. Thus, in order for g^{-1} to be a function, it would be necessary that g be injective.

But that is not enough. Function g also fails to be surjective because no element of A is sent to the element $1 \in B$. This means g^{-1} contains no ordered pair whose first coordinate is 1, so it can't be a function from B to A. If g^{-1} were to be a function it would be necessary that g be surjective.

The previous two paragraphs suggest that if g is a function, then it must be bijective in order for its inverse relation g^{-1} to be a function. Indeed, this is easy to verify. Conversely, if a function is bijective, then its inverse relation is easily seen to be a function. We summarize this in the following theorem.

Theorem 18.4. *Let $f: A \to B$ be a function. Then f is bijective if and only if the inverse relation f^{-1} is a function from B to A.*

Suppose $f: A \to B$ is bijective, so according to the theorem f^{-1} is a function. Observe that the relation f contains all the pairs $(x, f(x))$ for $x \in A$, so f^{-1} contains all the pairs $(f(x), x)$. But $(f(x), x) \in f^{-1}$ means $f^{-1}(f(x)) = x$. Therefore $f^{-1} \circ f\,(x) = x$ for every $x \in A$. From this we get $f^{-1} \circ f = i_A$. Similar reasoning produces $f \circ f^{-1} = i_B$. This leads to the following definitions.

Definition 18.8. If $f: A \to B$ is bijective then, its **inverse** is the function $f^{-1}: B \to A$. The functions f and f^{-1} obey $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

You probably recall from algebra a technique for finding the inverse of a bijective function f: to find f^{-1}, start with the equation $y = f(x)$. Then interchange variables to get $x = f(y)$. Solving this equation for y (if possible) produces $y = f^{-1}(x)$. The next two examples illustrate this.

Example 18.13. Find the inverse of the function $f: \mathbb{R} \to \mathbb{R}$, where $f(x) = x^3 + 1$.

Solution: We begin by writing $y = x^3 + 1$. Now interchange variables to obtain $x = y^3 + 1$. Solving for y produces $y = \sqrt[3]{x - 1}$. Thus

$$f^{-1}(x) = \sqrt[3]{x - 1}.$$

(You can check this by computing $f^{-1}(f(x)) = x$. $\sqrt[3]{f(x) - 1} = \sqrt[3]{x^3 + 1 - 1} = x$. Therefore $f^{-1}(f(x)) = x$. Any answer other than x indicates a mistake.) ✍

Example 18.14. Example 18.6 showed that the function $g\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ defined by the formula $g(m,n) = (m+n, m+2n)$ is bijective. Find its inverse.

Solution: The approach outlined above should work, but we need to be careful to keep track of coordinates in $\mathbb{Z}\times\mathbb{Z}$. We begin by writing $(x,y) = g(m,n)$, then interchanging the variables (x,y) and (m,n) to get $(m,n) = g(x,y)$. This gives

$$(m,n) = (x+y, x+2y),$$

from which we get the following system of equations:

$$\begin{aligned} x + y &= m \\ x + 2y &= n. \end{aligned}$$

Solving this system using methids from algebra with which you are familiar, we get

$$\begin{aligned} x &= 2m - n \\ y &= n - m. \end{aligned}$$

Then $(x,y) = (2m-n, n-m)$, so $\boxed{g^{-1}(m,n) = (2m-n, n-m).}$

We can check this by confirming that $g^{-1}(g(m,n)) = (m,n)$. Doing the math,

$$\begin{aligned} g^{-1}(g(m,n)) &= g^{-1}(m+n, m+2n) \\ &= \big(2(m+n) - (m+2n), (m+2n) - (m+n)\big) \\ &= (m,n). \end{aligned}$$

✍

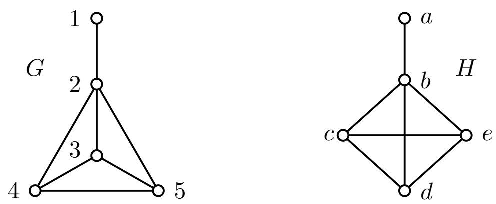
Exercises for Section 18.5

1. Check that $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 6 - n$ is bijective. Then find f^{-1}.
2. In Exercise 9 of Section 18.2 you proved that $f\colon \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \dfrac{5x+1}{x-2}$ is bijective. Now find its inverse.
3. Let $B = \{2^n : n \in \mathbb{Z}\} = \{\ldots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\}$. Show that the function $f\colon \mathbb{Z} \to B$ defined as $f(n) = 2^n$ is bijective. Then find f^{-1}.
4. The function $f\colon \mathbb{R} \to (0,\infty)$ defined as $f(x) = e^{x^3+1}$ is bijective. Find its inverse.
5. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \pi x - e$ is bijective. Find its inverse.
6. The function $f\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ defined by the formula $f(m,n) = (5m+4n, 4m+3n)$ is bijective. Find its inverse.
7. Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x,y) = ((x^2+1)y, x^3)$ is bijective. Then find its inverse.
8. Is the function $\theta : \mathscr{P}(\mathbb{Z}) \to \mathscr{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$ bijective? If so, what is its inverse?
9. Consider the function $f\colon \mathbb{R}\times\mathbb{N} \to \mathbb{N}\times\mathbb{R}$ defined as $f(x,y) = (y, 3xy)$. Check that this is bijective; find its inverse.
10. Consider $f\colon \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = \dfrac{(-1)^n(2n-1)+1}{4}$. This function is bijective by Exercise 18 in Section 18.2. Find its inverse.

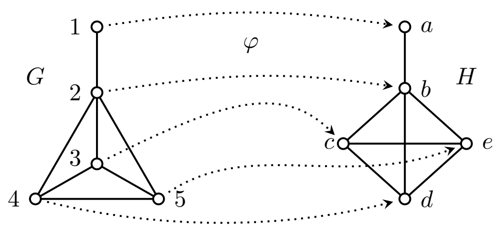
18.6 Case Study: Isomorphisms of Graphs

One very important feature of functions is that they provide a means of comparing mathematical structures. Almost every branch of mathematics uses functions in this capacity. In this section we concentrate on graph theory. We will see how functions determine and describe how two different graphs can have the same structure. (See Chapter 16 if you need to review graph notation and properties.)

To begin, consider the two graphs G and H shown below. The graph G has vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and edge set $E(G) = \{12, 23, 24, 25, 43, 35, 45\}$, whereas H has vertices $V(H) = \{a, b, c, d, e\}$ and edges $E(H) = \{ab, bc, bd, be, cd, ed, ce\}$.



Though G and H may look different, they have the same structure. This is highlighted by a bijective function $\varphi : V(G) \to V(H)$ indicated by dotted lines below. This function $\varphi = \{(1, a), (2, b), (3, c), (4, d), (5, e)\}$ has the property that $xy \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(H)$. (For instance, $12 \in E(G)$ and $\varphi(1)\varphi(2) = ab \in E(H)$, etc.) Thus φ describes how to overlay G on H so that they match perfectly.

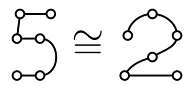


A function such as this one, that matches up G and H by sending vertices to vertices and edges to edges is called an *isomorphism*.

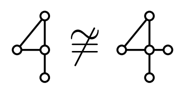
Definition 18.9. Two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are said to be **isomorphic** if there exists a bijective function $\varphi : V(G) \to V(H)$ such that $xy \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(H)$. Such a function φ is called an **isomorphism**. We express the condition of G and H being isomorphic as $G \cong H$. If G and H are *not* isomorphic, we write $G \ncong H$.

Though they may look different, two isomorphic graphs have identical structures, and are often thought of as being equal. The isomorphism tells how to match them.

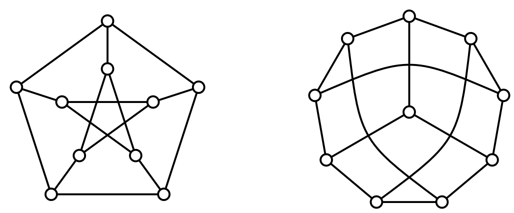
If the graphs are simple, you can usually spot immediately whether or not they are isomorphic without making a specific reference to the function φ. For example, the two graphs shown below are isomorphic, as they are both just paths of length 5.



But those shown below are *not* isomorphic. Because $|V(G)| \neq |V(H)|$, there can be no bijective function $\varphi : G \to H$, and hence no isomorphism.

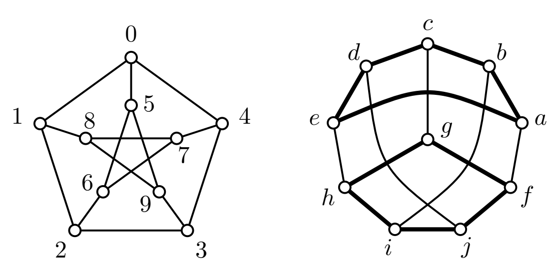


Example 18.15. Are the two graphs G and H shown below isomorphic?



G H

Solution: Let's begin with an informal analysis. The graph G consists of an outer pentagon and an inner pentagon (in the shape of a star) with radial edges connecting vertices of the two pentagons. But H has a similar description. The drawing below shows that H also has two pentagons (shown bold) with edges connecting them.



With the help of this visual aid, label the vertices of the two graphs as shown, and define φ as follows.

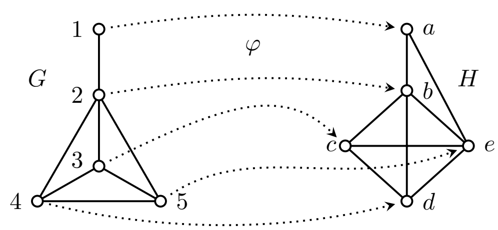
x	0	1	2	3	4	5	6	7	8	9
$\varphi(x)$	a	b	c	d	e	f	g	h	i	j

Check that $xy \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(H)$. (For example, $04 \in E(G)$ and $\varphi(0)\varphi(4) = ae \in E(H)$, etc.) Thus φ is an isomorphism, so $G \cong H$. ✍

Isomorphic graphs necessarily have the same number of vertices and the same number of edges. Further, any graph property possessed by one must be possessed by the other. Thus, for instance, if a bipartite graph G is isomorphic to some graph H, then H must also be bipartite. Conversely, if one graph is bipartite and another is not, then they cannot be isomorphic.

Read the definition of an isomorphism carefully. In order for $\varphi : V(G) \to V(H)$ to be an isomorphism, it is required that the function φ be bijective, *and* to satisfy $xy \in E(G)$ if and only if $\varphi(x)\varphi(y) \in E(H)$. If one of these conditions is not met, then φ is not an isomorphism.

For example, the φ below **is** bijective, and also $xy{\in}E(G) \Rightarrow \varphi(x)\varphi(y){\in}E(H)$. But φ is not an isomorphism because it's **not** true that $xy{\in}E(G){\Leftrightarrow}\varphi(x)\varphi(y){\in}E(H)$. In particular, $15 \notin E(G)$ but $\varphi(1)\varphi(5) = ac \in E(H)$.



But the fact that this particular function φ is not an isomorphism does not necessarily mean that $G \ncong H$, because perhaps there could be other functions $V(G) \to V(H)$ that *are* isomorphisms. But in this case we can say something definite: G and H are not isomorphic because they do not have the same number of edges.

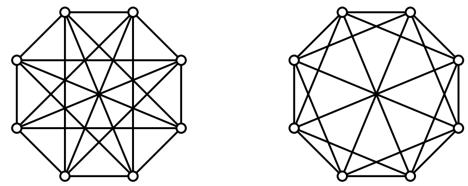
To show that two graphs are isomorphic, it suffices to exhibit an isomorphism between them. Showing that they are *not* isomorphic is often simpler: one needs only find a structural property possessed by one graph, but not the other. For example, above we noted that the G and H above are not isomorphic because G has 7 edges, while H has 8 edges. Alternatively we might have noted that G has no vertex of degree 2, but H does, or that G has no 5-cycle, but H does.

The relation $\cong$ is an equivalence relation on the set of all graphs. It is reflexive ($G \cong G$ for all graphs G), and it is symmetric ($G \cong H$ implies $H \cong G$). And it is transitive ($G \cong H$ and $H \cong K$ imply $G \cong K$). Exercise 6 asks for a verification.

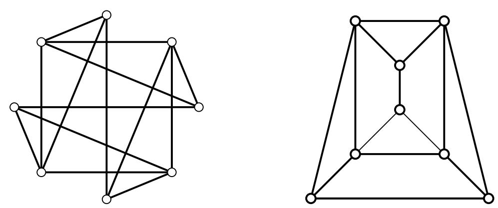
In. summary, graph isomorphism is a natural way of expressing when two graphs should be considered the same, or equal. Since equality of mathematical objects is such a fundaental notion, isomorphism plays an essential role in graph theory. Any further study of graph theory will involve this elemental idea.

Exercises for Section 18.6

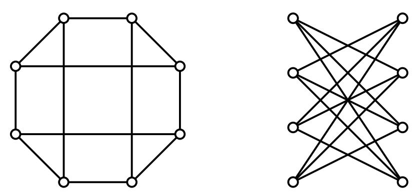
1. Decide if the two graphs below are isomorphic.



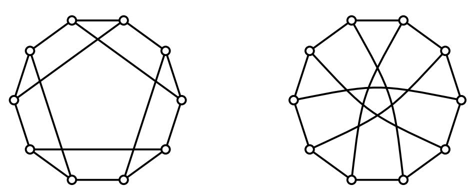
2. Decide if the two graphs below are isomorphic.



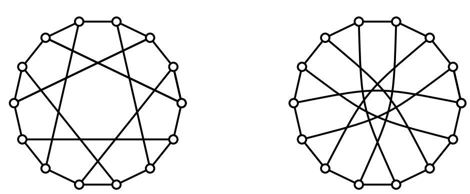
3. Decide if the two graphs below are isomorphic.



4. Decide if the two graphs below are isomorphic.



5. Decide if the two graphs below are isomorphic.



6. Show that $\cong$ is an equivalence relation on the set of all graphs.

Solutions for Chapter 18

Section 18.1 Exercises

1. Suppose $A = \{0,1,2,3,4\}$, $B = \{2,3,4,5\}$ and $f = \{(0,3),(1,3),(2,4),(3,2),(4,2)\}$. State the domain and range of f. Find $f(2)$ and $f(1)$.

Domain is A; Range is $\{2,3,4\}$; $f(2) = 4$; $f(1) = 3$.

3. There are four different functions $f\colon \{a,b\} \to \{0,1\}$. List them all.

$f_1 = \{(a,0),(b,0)\}$ $\quad f_2 = \{(a,1),(b,0)\}$, $\quad f_3 = \{(a,0),(b,1)\}$ $\quad f_4 = \{(a,1),(b,1)\}$

5. Give an example of a relation from $\{a,b,c,d\}$ to $\{d,e\}$ that is not a function.

One example is $\{(a,d),(a,e),(b,d),(c,d),(d,d)\}$.

7. Consider the set $f = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} : 3x+y=4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$?

Yes, since $3x+y=4$ if and only if $y = 4-3x$, this is the function $f\colon \mathbb{Z}\to\mathbb{Z}$ defined as $f(x) = 4-3x$.

9. Consider the set $f = \{(x^2,x) : x \in \mathbb{R}\}$. Is this a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

No. This is not a function because f contains the ordered pairs $(4,2)$ and $(4,-2)$. Thus the number 4 occurs as the first coordinate of more than one element of f.

11. Is the set $\theta = \{(X,|X|) : X \subseteq \mathbb{Z}_5\}$ a function? If so, what is its domain and range?

Yes, this is a function. The domain is $\mathscr{P}(\mathbb{Z}_5)$. The range is $\{0,1,2,3,4,5\}$.

Section 18.2 Exercises

1. Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$. Give an example of a function $f\colon A\to B$ that is neither injective nor surjective.

Consider $f = \{(1,a),(2,a),(3,a),(4,a)\}$. Then f is not injective because $f(1) = f(2)$. Also f is not surjective because it sends no element of A to the element $c \in B$.

3. Consider the cosine function $\cos : \mathbb{R}\to\mathbb{R}$. Decide whether this function is injective and whether it is surjective. What if it had been defined as $\cos : \mathbb{R}\to[-1,1]$?

The function $\cos : \mathbb{R}\to\mathbb{R}$ is **not injective** because, for example, $\cos(0) = \cos(2\pi)$. It is **not surjective** because if $b = 5 \in \mathbb{R}$ (for example), there is no real number for which $\cos(x) = b$. The function $\cos : \mathbb{R}\to[-1,1]$ **is surjective.** but not injective.

5. A function $f\colon \mathbb{Z}\to\mathbb{Z}$ is defined as $f(n) = 2n+1$. Verify whether this function is injective and whether it is surjective.

This function is injective. To see this, suppose $m,n \in \mathbb{Z}$ and $f(m) = f(n)$. Then $2m+1 = 2n+1$, from which we get $2m = 2n$, so then $m = n$. Thus f is injective.

This function is not surjective. To see this notice that $f(n)$ is odd for all $n \in \mathbb{Z}$. So given the (even) number 2 in the codomain $\mathbb{Z}$, there is no n with $f(n) = 2$.

7. A function $f\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ is defined as $f((m,n)) = 2n-4m$. Verify whether this function is injective and whether it is surjective.

This is **not injective** because $(0,2) \neq (-1,0)$, yet $f((0,2)) = f((-1,0)) = 4$. This is **not surjective** because $f((m,n)) = 2n-4m = 2(n-2m)$ is always even. If $b \in \mathbb{Z}$ is odd, then $f((m,n)) \neq b$, for all $(m,n) \in \mathbb{Z}\times\mathbb{Z}$.

9. Prove that the function $f\colon \mathbb{R}-\{2\}\to\mathbb{R}-\{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.

Proof. First, let's check that f is injective. Suppose $f(x) = f(y)$. Then

$$\begin{aligned} \frac{5x+1}{x-2} &= \frac{5y+1}{y-2} \\ (5x+1)(y-2) &= (5y+1)(x-2) \\ 5xy - 10x + y - 2 &= 5yx - 10y + x - 2 \\ -10x + y &= -10y + x \\ 11y &= 11x \\ y &= x. \end{aligned}$$

Since $f(x) = f(y)$ implies $x = y$, it follows that f is injective.

Next, let's check that f is surjective. For this, take an arbitrary element $b \in \mathbb{R} - \{5\}$. We want to see if there is an $x \in \mathbb{R} - \{2\}$ for which $f(x) = b$, or $\frac{5x+1}{x-2} = b$. Solving this for x, we get:

$$\begin{aligned} 5x + 1 &= b(x-2) \\ 5x + 1 &= bx - 2b \\ 5x - xb &= -2b - 1 \\ x(5-b) &= -2b - 1. \end{aligned}$$

Since we have assumed $b \in \mathbb{R} - \{5\}$, the term $(5-b)$ is not zero, and we can divide with impunity to get $x = \dfrac{-2b-1}{5-b}$. This is an x for which $f(x) = b$, so f is surjective. Since f is both injective and surjective, it is bijective. □

11. Consider the function $\theta : \{0,1\} \times \mathbb{N} \to \mathbb{Z}$ defined as $\theta(a,b) = (-1)^a b$. Is θ injective? Is it surjective? Explain.

First we show that θ is injective. Suppose $\theta(a,b) = \theta(c,d)$. Then $(-1)^a b = (-1)^c d$. As b and d are both in $\mathbb{N}$, they are both positive. Then because $(-1)^a b = (-1)^c d$, it follows that $(-1)^a$ and $(-1)^c$ have the same sign. Since each of $(-1)^a$ and $(-1)^c$ equals ± 1, we have $(-1)^a = (-1)^c$, so then $(-1)^a b = (-1)^c d$ implies $b = d$. But also $(-1)^a = (-1)^c$ means a and c have the same parity, and because $a, c \in \{0,1\}$, it follows $a = c$. Thus $(a,b) = (c,d)$, so θ is injective.

Next note that θ **is not surjective** because $\theta(a,b) = (-1)^a b$ is either positive or negative, but never zero. Therefore there exist no element $(a,b) \in \{0,1\} \times \mathbb{N}$ for which $\theta(a,b) = 0 \in \mathbb{Z}$.

13. Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x,y) = (xy, x^3)$. Is f injective? Is it surjective?

Notice that $f(0,1) = (0,0)$ and $f(0,0) = (0,0)$, so f is **not injective**. To show that f is also **not surjective**, we will show that it's impossible to find an ordered pair (x,y) with $f(x,y) = (1,0)$. If there were such a pair, then $f(x,y) = (xy, x^3) = (1,0)$, which yields $xy = 1$ and $x^3 = 0$. From $x^3 = 0$ we get $x = 0$, so $xy = 0$, a contradiction.

15. This question concerns functions $f \colon \{A,B,C,D,E,F,G\} \to \{1,2,3,4,5,6,7\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

Function f can described as a list $(f(A), f(B), f(C), f(D), f(E), f(F), f(G))$, where there are seven choices for each entry. By the multiplication principle, the total number of functions f is $7^7 = 823543$.

If f is injective, then this list can't have any repetition, so there are $7! = 5040$ injective functions. Since any injective function sends the seven elements of the domain to seven distinct elements of the codomain, all of the injective functions are surjective, and vice versa. Thus there are 5040 surjective functions and 5040 bijective functions.

17. This question concerns functions $f\colon \{A, B, C, D, E, F, G\} \to \{1, 2\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

Function f can described as a list $(f(A), f(B), f(C), f(D), f(E), f(F), f(G))$, where there are two choices for each entry. Therefore the total number of functions is $2^7 = 128$. It is impossible for any function to send all seven elements of $\{A, B, C, D, E, F, G\}$ to seven distinct elements of $\{1, 2\}$, so none of these 128 functions is injective, hence none are bijective.

How many are surjective? Only two of the 128 functions are not surjective, and they are the "constant" functions $\{(A, 1), (B, 1), (C, 1), (D, 1), (E, 1), (F, 1), (G, 1)\}$ and $\{(A, 2), (B, 2), (C, 2), (D, 2), (E, 2), (F, 2), (G, 2)\}$. So there are 126 surjective functions.

Section 18.3 Exercises

1. For any six integers, at least two have the same remainder when divided by 5.

Proof. Let A be a set of six integers and $B = \big\{0, 1, 2, 3, 4\big\}$. Define $f : A \to B$ so that $f(x)$ is the remainder when x is divided by 5. Because $|A| = 6 > 5 = |B|$, the pigeonhole principle guarantees that f is not injective. Thus there are two integers $x, y \in A$ for which $f(x) = f(y)$, meaning x and y have the same remainder when divided by 5. □

3. For any six integers, 9 divides the sum or difference of two of them.

Proof. Let A be a set of six integers. Let $B = \big\{ \{0\}, \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\} \big\}$. Notice that every element of B is a set that either has one element (0) or has two elements whose sum is 9. Define $f : A \to B$ so that $f(x)$ is the set in B that contains the remainder when x is divided by 9. For example, $f(12) = \{3, 6\}$ and $f(18) = \{0\}$. Since $6 = |A| > |B| = 5$, the pigeonhole principle implies that f is not injective. Thus there exist $x, y \in A$ for which $f(x) = f(y)$. Then either x and y both have the same reminder r when divided by 9, or the remainders r and s add to 9. In the first case $x = 9m + r$ and $y = 9n + r$ (for some $m, n \in \mathbb{Z}$), so 9 divides $x - y = 9(m - n)$. In the second case $x = 9m + r$ and $y = 9n + s$, so 9 divides $x + y = 9m + 9n + r + s = 9(m + n + 1)$. □

5. Any set of 7 integers contains a pair whose sum or difference is divisible by 10.

Proof. Let A be a set of 7 integers. Let $B = \big\{\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}, \{0\}\big\}$. So B is a set of six sets; in particular, $|B| = 6$. The four 2-elements sets in B contain numbers that add to 10. Let $f : A \to B$ be the function for which $f(x)$ equals the set in B that contains the remainder when x is divided by 10. (Examples: $f(97) = \{3, 7\}$, $f(13) = \{3, 7\}$, $f(12) = \{2, 8\}$, $f(230) = \{0\}$, $f(15) = \{5\}$, etc.) Because $|A| > |B|$, the pigeonhole principle guarantees that f is not injective. Select two integers $x, y \in A$ for which $f(x) = f(y)$. If x and y happen to have the same remainder when divided by 10, then their difference $x - y$ is divisible by 10. If x and y and don't have the same remainder when divided by 10, then $f(x) = f(y) = \{r, s\}$ is one of the 2-element sets in B. In this case, $x = 10k+r$ and $y = 10\ell+s$ (for integers k and ℓ), while $r+s = 10$. Then the sum $x + y = (10k + r) + (10\ell + s) = 10k + 10\ell + 10 = 10(k + \ell + 1)$ is divisible by 10. □

7. If $X \subseteq \{1, 2, 3, \ldots, 2n\}$ and $|X| > n$, then one element of X divides another.

Proof. Say X is as stated, and let $Y = \{1, 3, 5, \ldots, 2n-1\}$ be the set of positive odd integers less than $2n$; note $|Y| = n$. Any positive integer m can be factored as $m = 2^p q$ where q is the largest odd integer dividing m. For example, $100 = 2^2 25$, $12 = 2^2 3$, $8 = 2^3 1$ and $13 = 3^0 13$. Let $f : X \to Y$ be such that $f(m)$ equals the largest odd integer dividing m. For example, $f(100) = 25$, $f(12) = 3$, $f(8) = 1$ and $f(13) = 13$. Because $|X| > |Y|$, the pigeonhole principle guarantees f is not injective. Thus there are two integers $a, b \in X$ for which $f(a) = f(b)$. Say $f(a) = q$. Then $a = 2^p q$ and $b = 2^r q$ for some p and r. If $p < r$ then $a \mid b$. If $r < p$, then $b \mid a$. □

Section 18.4 Exercises

1. Suppose $A = \{5, 6, 8\}$, $B = \{0, 1\}$, $C = \{1, 2, 3\}$. Let $f\colon A \to B$ be the function $f = \{(5, 1), (6, 0), (8, 1)\}$, and $g\colon B \to C$ be $g = \{(0, 1), (1, 1)\}$. Find $g \circ f$.

$g \circ f = \{(5, 1), (6, 1), (8, 1)\}$

3. Suppose $A = \{1, 2, 3\}$. Let $f\colon A \to A$ be the function $f = \{(1, 2), (2, 2), (3, 1)\}$, and let $g\colon A \to A$ be the function $g = \{(1, 3), (2, 1), (3, 2)\}$. Find $g \circ f$ and $f \circ g$.

$g \circ f = \{(1, 1), (2, 1), (3, 3)\}$; $f \circ g = \{(1, 1), (2, 2), (3, 2)\}$.

5. Consider the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sqrt[3]{x+1}$ and $g(x) = x^3$. Find the formulas for $g \circ f$ and $f \circ g$.

$g \circ f\,(x) = x + 1$; $f \circ g\,(x) = \sqrt[3]{x^3 + 1}$

7. Consider the functions $f, g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (mn, m^2)$ and $g(m, n) = (m + 1, m + n)$. Find the formulas for $g \circ f$ and $f \circ g$.

Note $g \circ f\,(m, n) = g(f(m, n)) = g(mn, m^2) = (mn + 1, mn + m^2)$.

Thus $\boxed{g \circ f\,(m, n) = (mn + 1, mn + m^2).}$

Note $f \circ g\,(m, n) = f(g(m, n)) = f(m + 1, m + n) = ((m + 1)(m + n), (m + 1)^2)$.

Thus $\boxed{f \circ g\,(m, n) = (m^2 + mn + m + n, m^2 + 2m + 1).}$

9. Consider the functions $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $f(m, n) = m + n$ and $g\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $g(m) = (m, m)$. Find the formulas for $g \circ f$ and $f \circ g$.

$g \circ f(m, n) = (m + n, m + n)$
$f \circ g(m) = 2m$

Section 18.5 Exercises

1. Check that the function $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 6 - n$ is bijective. Find f^{-1}.

It is injective as follows. Suppose $f(m) = f(n)$. Then $6 - m = 6 - n$, which reduces to $m = n$.
It is surjective as follows. If $b \in \mathbb{Z}$, then $f(6 - b) = 6 - (6 - b) = b$.
Inverse: $f^{-1}(n) = 6 - n$.

3. Let $B = \{2^n : n \in \mathbb{Z}\} = \left\{\ldots, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, \ldots\right\}$. Show that the function $f\colon \mathbb{Z} \to B$ defined as $f(n) = 2^n$ is bijective. Then find f^{-1}.

It is injective as follows. Suppose $f(m) = f(n)$, which means $2^m = 2^n$. Taking $\log_2$ of both sides gives $\log_2(2^m) = \log_2(2^n)$, which simplifies to $m = n$.
The function f is surjective as follows. Suppose $b \in B$. By definition of B this means $b = 2^n$ for some $n \in \mathbb{Z}$. Then $f(n) = 2^n = b$.
Inverse: $f^{-1}(n) = \log_2(n)$.

5. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \pi x - e$ is bijective. Find its inverse.

Inverse: $f^{-1}(x) = \dfrac{x+e}{\pi}$.

7. Show that the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f((x, y) = ((x^2+1)y, x^3)$ is bijective. Then find its inverse.

First we prove the function is injective. Assume $f(x_1, y_1) = f(x_2, y_2)$. Then $(x_1^2 + 1)y_1 = (x_2^2+1)y_2$ and $x_1^3 = x_2^3$. Since the real-valued function $f(x) = x^3$ is one-to-one, it follows that $x_1 = x_2$. Since $x_1 = x_2$, and $x_1^2 + 1 > 0$ we may divide both sides of $(x_1^2 + 1)y_1 = (x_1^2 + 1)y_2$ by $(x_1^2 + 1)$ to get $y_1 = y_2$. Hence $(x_1, y_1) = (x_2, y_2)$.

Now we prove the function is surjective. Let $(a, b) \in \mathbb{R}^2$. Set $x = b^{1/3}$ and $y = a/(b^{2/3}+1)$. Then $f(x, y) = ((b^{2/3}+1)\frac{a}{b^{2/3}+1}, (b^{1/3})^3) = (a, b)$. It now follows that f is bijective.

Finally, we compute the inverse. Write $f(x, y) = (u, v)$. Interchange variables to get $(x, y) = f(u, v) = ((u^2 + 1)v, u^3)$. Thus $x = (u^2 + 1)v$ and $y = u^3$. Hence $u = y^{1/3}$ and $v = \frac{x}{y^{2/3}+1}$. Therefore $f^{-1}(x, y) = (u, v) = \left(y^{1/3}, \frac{x}{y^{2/3}+1}\right)$.

9. Consider the function $f\colon \mathbb{R} \times \mathbb{N} \to \mathbb{N} \times \mathbb{R}$ defined as $f(x, y) = (y, 3xy)$. Check that this is bijective; find its inverse.

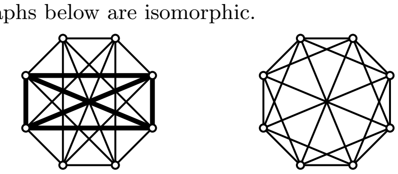
To see that this is injective, suppose $f(a, b) = f(c, d)$. This means $(b, 3ab) = (d, 3cd)$. Since the first coordinates must be equal, we get $b = d$. As the second coordinates are equal, we get $3ab = 3dc$, which becomes $3ab = 3bc$. Note that, from the definition of f, $b \in \mathbb{N}$, so $b \neq 0$. Thus we can divide both sides of $3ab = 3bc$ by the non-zero quantity $3b$ to get $a = c$. Now we have $a = c$ and $b = d$, so $(a, b) = (c, d)$. It follows that f is injective.

Next we check that f is surjective. Given any (b, c) in the codomain $\mathbb{N} \times \mathbb{R}$, notice that $(\frac{c}{3b}, b)$ belongs to the domain $\mathbb{R} \times \mathbb{N}$, and $f(\frac{c}{3b}, b) = (b, c)$. Thus f is surjective. As it is both injective and surjective, it is bijective; thus the inverse exists.

To find the inverse, recall that we obtained $f(\frac{c}{3b}, b) = (b, c)$. Then $f^{-1}f(\frac{c}{3b}, b) = f^{-1}(b, c)$, which reduces to $(\frac{c}{3b}, b) = f^{-1}(b, c)$. Replacing b and c with x and y, respectively, we get $f^{-1}(x, y) = (\frac{y}{3x}, x)$.

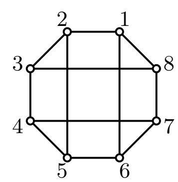
Exercises for Section 18.6

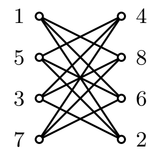
1. Decide if the two graphs below are isomorphic.



They are not isomorphic. The one on the left has a K_4 subgraph (shown bold), but the one on the right does not.

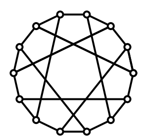
3. Decide if the two graphs below are isomorphic.

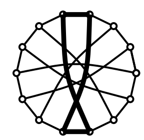




They are isomorphic. With the labeling above, check that the function $\varphi(x) = x$ is an isomorphism.

5. Decide if the two graphs below are isomorphic.





They are not isomorphic. The one on the right has 4-cycles (one is shown bold) but the one on the left has none.