

Chapter 3

Logic

Logic is a systematic way of thinking that is profoundly important to mathematics. In this book logic has five main purposes. (1) It allows us to correctly deduce new information from old information; (2) it gives precise meanings to certain words and linguistic constructions that occur in mathematics; (3) it underlies all strategies for proving mathematical statements; (4) it controls the flow of computer programs; and (5) it is the foundation of the design of computer circuitry.

You already use logic informally in everyday life and certainly also in doing mathematics. For example, suppose you are working with a certain circle, call it “Circle X,” and you have available the following two pieces of information.

1. Circle X has radius equal to 3 units.
2. If any circle has radius r , then its area is πr^2 square units.

You have no trouble putting these two facts together to get:

3. Circle X has area 9π square units.

In doing this you are using logic to combine existing information to produce new information. Because deducing new information is central to mathematics, logic plays a fundamental role.

It is important to realize that logic is a process of deducing information correctly, *not* just deducing correct information. For example, suppose we were mistaken and Circle X actually had radius 4, not 3. Let’s look at our exact same argument again.

1. Circle X has radius equal to 3.
 2. If any circle has radius r , then its area is πr^2 square units.
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3. Circle X has area 9π square units.

The sentence “*Circle X has radius equal to 3.*” is now untrue, and so is our conclusion “*Circle X has area 9π square units.*” But the logic is perfectly correct; the information was combined correctly, even if some of it was false. This distinction between correct logic and correct information is significant because it is often important to follow the consequences of an incorrect assumption. Ideally, we want both our logic *and* our information to be correct, but the point is that they are different things.

The study of logic begins with *statements*.

3.1 Statements

A **statement** is a sentence or a mathematical expression that is either definitely true or definitely false. You can think of statements as pieces of information that are either correct or incorrect. Thus statements are pieces of information that we might apply logic to in order to produce other pieces of information (which are also statements).

Example 3.1. Here are some examples of statements. They are all true.

If a circle has radius r , then its area is πr^2 square units.


Every even number is divisible by 2.

$2 \in \mathbb{Z}$

$\sqrt{2} \notin \mathbb{Z}$

$\mathbb{N} \subseteq \mathbb{Z}$

The set $\{0, 1, 2\}$ has three elements.

Some right triangles are isosceles. 


Example 3.2. Here are some additional statements. They are all false.

All right triangles are isosceles.

$5 = 2$

$\sqrt{2} \notin \mathbb{R}$

$\mathbb{Z} \subseteq \mathbb{N}$

$\{0, 1, 2\} \cap \mathbb{N} = \emptyset$ 

Example 3.3. Here we pair sentences or expressions that are not statements with similar expressions that *are* statements.

NOT Statements:	Statements:
Add 5 to both sides.	Adding 5 to both sides of $x - 5 = 37$ gives $x = 42$.
\mathbb{Z}	$42 \in \mathbb{Z}$
42	42 is not a number.
What is the solution of $2x = 84$?	The solution of $2x = 84$ is 42.


Example 3.4. We use the letters P , Q , R and S to stand for specific statements. When more letters are needed we can use subscripts. Here are more statements, designated with letters. You decide which of them are true and which are false.

P : For every integer $n > 1$, the number $2^n - 1$ is prime.

Q : Every polynomial of degree n has at most n roots.

R : The function $f(x) = x^2$ is continuous.

S_1 : $\mathbb{Z} \subseteq \emptyset$

S_2 : $\{0, -1, -2\} \cap \mathbb{N} = \emptyset$ 

Designating statements with letters (as was done above) is a very useful shorthand. In discussing a particular statement, such as “The set $\{a, b, c\}$ has three elements,” it is convenient to just refer to it as R to avoid having to write or say it many times.

Statements can contain variables. Here is an example.

P : If an integer x is a multiple of 6, then x is even.

This is a sentence that is true. (All multiples of 6 are even, so no matter which multiple of 6 the integer x happens to be, it is even.) Since the sentence P is definitely true, it is a statement. When a sentence or statement P contains a variable such as x , we sometimes denote it as $P(x)$ to indicate that it is saying something about x . Thus the above statement can be denoted as

$P(x)$: If an integer x is a multiple of 6, then x is even.

A statement or sentence involving two variables might be denoted $P(x, y)$, etc.

It is quite possible for a sentence containing variables to not be a statement. Consider the following example.

$Q(x)$: The integer x is even.

Is this a statement? Whether it is true or false depends on just which integer x is. It is true if $x = 4$ and false if $x = 7$, etc. But without any stipulations on the value of x it is impossible to say whether $Q(x)$ is true or false. Since it is neither definitely true nor definitely false, $Q(x)$ is not a statement. A sentence such as this, whose truth depends on the value of one or more variables, is called an **open sentence**. The variables in an open sentence (or statement) can represent any type of entity, not just numbers. Here is an open sentence where the variables are functions:

$R(f, g)$: The function f is the derivative of the function g .

This open sentence is true if $f(x) = 2x$ and $g(x) = x^2$. It is false if $f(x) = x^3$ and $g(x) = x^2$, etc. We point out that a sentence such as $R(f, g)$ (that involves variables) can be denoted either as $R(f, g)$ or just R . We use the expression $R(f, g)$ when we want to emphasize that the sentence involves variables.

We say more about open sentences later. For now let's return to statements.

Statements are everywhere in mathematics. Anything that has been proved true is a statement. The quadratic formula and Pythagorean theorem are statements:

P : The solutions of the equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Q : If a right triangle has leg lengths a and b and hypotenuse c , then $a^2 + b^2 = c^2$.

Here is a very famous statement, so famous, in fact, that it has a name. It is called **Fermat's last theorem** after Pierre Fermat, a seventeenth-century French mathematician who scribbled it in the margin of a notebook.

R : For all numbers $a, b, c, n \in \mathbb{N}$ with $n > 2$, it is the case that $a^n + b^n \neq c^n$.

Fermat believed this statement was true. He noted that he could prove it was true, except his notebook's margin was too narrow to contain his proof. It is doubtful that he really had a correct proof in mind, for after his death generations of brilliant mathematicians tried unsuccessfully to prove that his statement was true (or false). Finally, in 1993, Andrew Wiles of Princeton University announced that he had devised a proof. Wiles had worked on the problem for over seven years. The moral of this story is that some true statements are not obviously true.

Here is another statement famous enough to be named. It was first posed in the eighteenth century by the German mathematician Christian Goldbach, and thus is called the **Goldbach conjecture**:

S : Every even integer greater than 2 is a sum of two prime numbers.

You must agree that S is either true or false. It appears to be true, because when you examine even numbers that are bigger than 2, they seem to be sums of two primes: $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 5 + 5$, $12 = 5 + 7$, $100 = 17 + 83$ and so on. But that's not to say there isn't some large even number that's not the sum of two primes. If such a number exists, then S is false. The thing is, in the over 260 years since Goldbach first posed this problem, no one has been able to determine whether it's true or false. But since it is clearly either true or false, S is a statement.

Much of this book is about the methods that can be used to prove that S (or any other statement) is true or false. To prove that a statement is true, we start with obvious statements (or other statements that have been proven true) and use logic to deduce more and more complex statements until finally we obtain a statement such as S . Of course some statements are more difficult to prove than others, and S appears to be notoriously difficult; we will concentrate on statements that are easier to prove.

But the point is this: In proving that statements are true, we use logic to help us understand statements and to combine pieces of information to produce new information. In the next several sections we explore some standard ways that statements can be combined to form new statements, or broken down into simpler ones.

Exercises for Section 3.1

Decide whether or not the following are statements. In the case of a statement, say if it is true or false, if possible.

1. Every real number is an even integer.
2. Every even integer is a real number.
3. If x and y are real numbers and $5x = 5y$, then $x = y$.
4. Sets \mathbb{Z} and \mathbb{N} .
5. Sets \mathbb{Z} and \mathbb{N} are infinite.
6. Some sets are finite.
7. The derivative of any polynomial of degree 5 is a polynomial of degree 6.
8. $\mathbb{N} \notin \mathcal{P}(\mathbb{N})$.
9. $\cos(x) = -1$
10. $(\mathbb{R} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{R}) = \mathbb{N} \times \mathbb{N}$
11. The integer x is a multiple of 7.
12. If the integer x is a multiple of 7, then it is divisible by 7.
13. Either x is a multiple of 7, or it is not.
14. Call me Ishmael.
15. In the beginning, God created the heaven and the earth.

3.2 And, Or, Not

The word “and” can be used to combine two statements to form a new statement. Consider for example the following sentence.

R_1 : The number 2 is even **and** the number 3 is odd.

We recognize this as a true statement, based on our common-sense understanding of the meaning of the word “and.” Notice that R_1 is made up of two simpler statements:

P : The number 2 is even.

Q : The number 3 is odd.

These are joined together by the word “and” to form the more complex statement R_1 . The statement R_1 asserts that P and Q are both true. Since both P and Q are in fact true, the statement R_1 is also true.

Had one or both of P and Q been false, then R_1 would be false. For instance, each of the following statements is false.

R_2 : The number 1 is even **and** the number 3 is odd.

R_3 : The number 2 is even **and** the number 4 is odd.

R_4 : The number 3 is even **and** the number 2 is odd.

From these examples we see that any two statements P and Q can be combined to form a new statement “ P **and** Q .” In the spirit of using letters to denote statements, we now introduce the special symbol \wedge to stand for the word “and.” Thus if P and Q are statements, $P \wedge Q$ stands for the statement “ P **and** Q .” The statement $P \wedge Q$ is true if both P and Q are true; otherwise it is false. This is summarized in the following table, called a **truth table**.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

In this table, T stands for “True,” and F stands for “False.” (T and F are called **truth values**.) Each line lists one of the four possible combinations or truth values for P and Q , and the column headed by $P \wedge Q$ tells whether the statement $P \wedge Q$ is true or false in each case.

Statements can be combined using the word “or.” Consider the following statements.

S_1 : The number 2 is even **or** the number 3 is odd.

S_2 : The number 1 is even **or** the number 3 is odd.

S_3 : The number 2 is even **or** the number 4 is odd.

S_4 : The number 3 is even **or** the number 2 is odd.

In mathematics, the assertion “ P **or** Q ” is always understood to mean that one *or both* of P and Q is true. Thus statements S_1 , S_2 , S_3 are all true, while S_4 is false.

The symbol \vee is used to stand for the word “or.” So if P and Q are statements, $P \vee Q$ represents the statement “ P or Q .” Here is the truth table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

It is important to be aware that the meaning of “or” expressed in the above table differs from the way it is often used in everyday conversation. For example, suppose a university official makes the following threat:

You pay your tuition **or** you will be withdrawn from school.

You understand that this means that either you pay your tuition *or* you will be withdrawn from school, *but not both*. In mathematics we never use the word “or” in such a sense. For us “or” means exactly what is stated in the table for \vee . Thus $P \vee Q$ being true means *one or both* of P and Q is true. If we ever need to express the fact that exactly one of P and Q is true, we use one of the following constructions:

P or Q , but not both.

Either P or Q .

Exactly one of P or Q .

If the university official were a mathematician, he might have qualified his statement in one of the following ways.

Pay your tuition **or** you will be withdrawn from school, **but not both**.

Either you pay your tuition **or** you will be withdrawn from school.

To conclude this section, we mention another way of obtaining new statements from old ones. Given any statement P , we can form the new statement “**It is not true that P .**” For example, consider the following statement.

The number 2 is even.

This is true. Now insert the words “It is not true that” at the beginning:

It is not true that the number 2 is even.

This new statement is false.

For another example, starting with the false statement “ $2 \in \emptyset$,” we get the true statement “It is not true that $2 \in \emptyset$.”

We use the symbol \neg to stand for the words “It’s not true that,” so $\neg P$ means “**It’s not true that P .**” We often read $\neg P$ simply as “not P .” Unlike \wedge and \vee , which combine two statements, the symbol \neg just alters a single statement. Thus its truth table has just two lines, one for each possible truth value of P .

P	$\neg P$
T	F
F	T

The statement $\neg P$ is called the **negation** of P .

The negation of a specific statement can be expressed in many ways. Consider

P : The number 2 is even.

Here are several ways of expressing its negation.

$\neg P$: It's not true that the number 2 is even.

$\neg P$: It is false that the number 2 is even.

$\neg P$: The number 2 is not even.

In this section we've learned how to combine or modify statements with the operations \wedge , \vee and \neg . Of course we can also apply these operations to open sentences or a mixture of open sentences and statements. For example, $(x \text{ is an even integer}) \wedge (3 \text{ is an odd integer})$ is an open sentence that is a combination of an open sentence and a statement.

Exercises for Section 3.2

Express each statement or open sentence in one of the forms $P \wedge Q$, $P \vee Q$, or $\neg P$. Be sure to also state exactly what statements P and Q stand for.

1. The number 8 is both even and a power of 2.
2. The matrix A is not invertible.
3. $x \neq y$
4. $x < y$
5. $y \geq x$
6. There is a quiz scheduled for Wednesday or Friday.
7. The number x equals zero, but the number y does not.
8. At least one of the numbers x and y equals 0.
9. $x \in A - B$
10. $x \in A \cup B$
11. $A \in \{X \in \mathcal{P}(\mathbb{N}) : |\overline{X}| < \infty\}$
12. Happy families are all alike, but each unhappy family is unhappy in its own way. (Leo Tolstoy, *Anna Karenina*)
13. Human beings want to be good, but not too good, and not all the time. (George Orwell)
14. A man should look for what is, and not for what he thinks should be. (Albert Einstein)

3.3 Conditional Statements

There is yet another way to combine two statements. Suppose we have in mind a specific integer a . Consider the following statement about a .

R : If the integer a is a multiple of 6, then a is divisible by 2.

We immediately spot this as a true statement based on our knowledge of integers and the meanings of the words "if" and "then." If integer a is a multiple of 6, then a is even, so therefore a is divisible by 2. Notice that R is built up from two simpler statements:

P : The integer a is a multiple of 6.

Q : The integer a is divisible by 2.

R : If P , then Q .

In general, given any two statements P and Q whatsoever, we can form the new statement “If P , then Q .” This is written symbolically as $P \Rightarrow Q$ which we read as “If P , then Q ,” or “ P implies Q .” Like \wedge and \vee , the symbol \Rightarrow has a very specific meaning. When we assert that the statement $P \Rightarrow Q$ is true, we mean that *if* P is true *then* Q must also be true. (In other words we mean that the condition P being true forces Q to be true.) A statement of form $P \Rightarrow Q$ is called a **conditional** statement because it means Q will be true *under the condition* that P is true.

Think of $P \Rightarrow Q$ as a promise that whenever P is true, Q will be true also. There is only one way this promise can be broken (i.e. be false) and that is if P is true but Q is false. Thus the truth table for the promise $P \Rightarrow Q$ is this:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Perhaps you are bothered by the fact that $P \Rightarrow Q$ is true in the last two lines of this table. Here’s an example to convince you that the table is correct. Suppose your professor makes the following promise:

If you pass the final exam, **then** you will pass the course.

(You pass the exam) \Rightarrow (You pass the course).

Under what circumstances did she lie? There are four possible scenarios, depending on whether or not you passed the exam and whether or not you passed the course.

You pass exam	You pass course	(You pass exam) \Rightarrow (You pass course)
T	T	T
T	F	F
F	T	T
F	F	T

In the first line you pass the exam and you pass the course. Your professor kept her promise, and the T in the third column indicates she told the truth. In the second line, you passed the exam, but your professor gave you a failing grade in the course. In this case she broke her promise, and the F in the third column indicates that what she said was untrue.

Now consider the third row. Here you failed the exam but passed the course. How could that happen? Maybe your professor felt sorry for you. But that doesn’t make her a liar. Her only promise was that if you passed the exam then you would pass the course. She did not say passing the exam was the *only way* to pass the course. She didn’t lie, so she told the truth, so there is a T in the third column.

Finally look at the fourth row. In that scenario you failed the exam and you failed the course. Your professor did not lie; she did exactly what she said she would do. Hence the T in the third column.

Here is another example that explains why $P \Rightarrow Q$ is true whenever P is false. Consider the following statement.

If this month is September, **then** there is an equinox this month.

An *equinox* is a day for which there are equal hours of darkness and light. There are two equinoxes per year, one in September and the other in March. The above statement is thus unquestionably true, for it asserts correctly that if the current month is September, then an equinox will occur this month. In symbolic form, our statement is

(This month is September) \Rightarrow (There is an equinox this month).

This statement is always true, no matter the month in which we say it. It is true if we say it in September, and it is true if we say it in March, or May, or any other month. But the open sentences P : “*This month is September,*” and Q : “*There is an equinox this month,*” are either true or false, depending on what month it is. Still, $P \Rightarrow Q$ is always true. This is tallied below for five months. Remember that in this example $P \Rightarrow Q$ is always true, and notice how this can be so even when P is false.

	This month is September	There is an equinox this month	$\left(\text{This month is September} \right) \Rightarrow \left(\text{There is an equinox this month} \right)$
Sept.	<i>T</i>	<i>T</i>	<i>T</i>
Oct.	<i>F</i>	<i>F</i>	<i>T</i>
Nov.	<i>F</i>	<i>F</i>	<i>T</i>
Dec.	<i>F</i>	<i>F</i>	<i>T</i>
March	<i>F</i>	<i>T</i>	<i>T</i>

As $P \Rightarrow Q$ is a true statement in this particular example, there is no month with P true and Q false. (Unless we imagine that Earth is destroyed by an asteroid before September 21, a possibility that we shall not entertain.)

Here is a summary of what we have learned so far about conditional statements.

The truth table for $P \Rightarrow Q$ is shown on the right. In mathematics, the sentence “*If P, then Q*” means exactly what the truth table for $P \Rightarrow Q$ expresses. It promises that P being true will make Q true too. So the only way $P \Rightarrow Q$ can be false is if P is true and Q is false. In particular, if P is false, then $P \Rightarrow Q$ is true.

P	Q	$P \Rightarrow Q$
<i>T</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>T</i>

Of course there are other grammatical constructions that mean exactly the same thing as $P \Rightarrow Q$. Here is a summary of the main ones.

$$\left. \begin{array}{l} \text{If } P, \text{ then } Q. \\ Q \text{ if } P. \\ Q \text{ whenever } P. \\ Q, \text{ provided that } P. \\ \text{Whenever } P, \text{ then also } Q. \\ P \text{ is a sufficient condition for } Q. \\ \text{For } Q, \text{ it is sufficient that } P. \\ Q \text{ is a necessary condition for } P. \\ \text{For } P, \text{ it is necessary that } Q. \\ P \text{ only if } Q. \end{array} \right\} P \Rightarrow Q$$

These can all be used in the place of (and mean exactly the same thing as) “*If P , then Q .*” You should analyze the meaning of each one and convince yourself that it captures the meaning of $P \Rightarrow Q$. For example, $P \Rightarrow Q$ means the condition of P being true is enough (i.e., sufficient) to make Q true; hence “ *P is a sufficient condition for Q .*”

The wording can be tricky. Often an everyday situation involving a conditional statement can help clarify it. For example, consider your professor’s promise:

$$(\text{You pass the exam}) \Rightarrow (\text{You pass the course})$$

This means that your passing the exam is a sufficient (though perhaps not necessary) condition for your passing the course. Thus your professor might just as well have phrased her promise in one of the following ways.

Passing the exam is a sufficient condition for passing the course.

For you to pass the course, it is sufficient that you pass the exam.

However, when we want to say “*If P , then Q* ” in everyday conversation, we do not normally express this as “ *Q is a necessary condition for P* ” or “ *P only if Q* .” But such constructions are not uncommon in mathematics. To understand why they make sense, notice that $P \Rightarrow Q$ being true means that it’s impossible that P is true but Q is false, so in order for P to be true it is necessary that Q is true; hence “ *Q is a necessary condition for P* .” And this means that P can only be true if Q is true, i.e., “ *P only if Q* .”

Exercises for Section 3.3

Without changing their meanings, convert each of the following sentences into a sentence having the form “*If P , then Q* .”

1. A matrix is invertible provided that its determinant is not zero.
2. For a function to be continuous, it is sufficient that it is differentiable.
3. For a function to be integrable, it is necessary that it is continuous.
4. A function is rational if it is a polynomial.
5. An integer is divisible by 8 only if it is divisible by 4.
6. Whenever a surface has only one side, it is non-orientable.

7. A series converges whenever it converges absolutely.
8. A geometric series with ratio r converges if $|r| < 1$.
9. A function is integrable provided the function is continuous.
10. The discriminant is negative only if the quadratic equation has no real solutions.
11. You fail only if you stop writing. (Ray Bradbury)
12. People will generally accept facts as truth only if the facts agree with what they already believe. (Andy Rooney)
13. Whenever people agree with me I feel I must be wrong. (Oscar Wilde)

3.4 Biconditional Statements

It is important to understand that $P \Rightarrow Q$ is not the same as $Q \Rightarrow P$. To see why, suppose that a is some integer and consider the statements

$$\begin{aligned}(a \text{ is a multiple of } 6) &\Rightarrow (a \text{ is divisible by } 2), \\ (a \text{ is divisible by } 2) &\Rightarrow (a \text{ is a multiple of } 6).\end{aligned}$$

The first statement asserts that if a is a multiple of 6 then a is divisible by 2. This is clearly true, for any multiple of 6 is even and therefore divisible by 2. The second statement asserts that if a is divisible by 2 then it is a multiple of 6. This is not necessarily true, for $a = 4$ (for instance) is divisible by 2, yet not a multiple of 6. Therefore the meanings of $P \Rightarrow Q$ and $Q \Rightarrow P$ are in general quite different. The conditional statement $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$, so a conditional statement and its converse express entirely different things.

But sometimes, if P and Q are just the right statements, it can happen that $P \Rightarrow Q$ and $Q \Rightarrow P$ are both necessarily true. For example, consider the statements

$$\begin{aligned}(a \text{ is even}) &\Rightarrow (a \text{ is divisible by } 2), \\ (a \text{ is divisible by } 2) &\Rightarrow (a \text{ is even}).\end{aligned}$$

No matter what value a has, both of these statements are true. Since both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, it follows that $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is true.

We now introduce a new symbol \Leftrightarrow to express the meaning of the statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. The expression $P \Leftrightarrow Q$ is understood to have exactly the same meaning as $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. According to the previous section, $Q \Rightarrow P$ is read as “ P if Q ,” and $P \Rightarrow Q$ can be read as “ P only if Q .” Therefore we pronounce $P \Leftrightarrow Q$ as “ P if and only if Q .” For example, given an integer a , we have the true statement

$$(a \text{ is even}) \Leftrightarrow (a \text{ is divisible by } 2),$$

which we can read as “*Integer a is even if and only if a is divisible by 2.*”

The truth table for \Leftrightarrow is shown below. Notice that in the first and last rows, both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true (by the truth table for \Rightarrow), so $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is true, and hence $P \Leftrightarrow Q$ is true. However, in the middle two rows one of $P \Rightarrow Q$ or $Q \Rightarrow P$ is false, so $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is false, making $P \Leftrightarrow Q$ false.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Compare the statement $R : (a \text{ is even}) \Leftrightarrow (a \text{ is divisible by } 2)$ with this truth table. If a is even then the two statements on either side of \Leftrightarrow are true, so the table says R is true. If a is odd then the two statements on either side of \Leftrightarrow are false, and again according to the table R is true. Thus R is true no matter what value a has. In general, $P \Leftrightarrow Q$ being true means P and Q are both true or both false.

Not surprisingly, there are many ways of saying $P \Leftrightarrow Q$ in English. The following constructions all mean $P \Leftrightarrow Q$:

$$\left. \begin{array}{l} P \text{ if and only if } Q. \\ P \text{ is a necessary and sufficient condition for } Q. \\ \text{For } P \text{ it is necessary and sufficient that } Q. \\ \text{If } P, \text{ then } Q, \text{ and conversely.} \\ \text{If } Q, \text{ then } P, \text{ and conversely.} \end{array} \right\} P \Leftrightarrow Q$$

The first three of these just combine constructions from the previous section to express that $P \Rightarrow Q$ and $Q \Rightarrow P$. In the fourth one, the words “...and conversely” mean that in addition to “If P , then Q ” being true, the converse statement “If Q , then P ” is also true.

Exercises for Section 3.4

Without changing their meanings, convert each of the following sentences into a sentence having the form “ P if and only if Q .”

1. For matrix A to be invertible, it is necessary and sufficient that $\det(A) \neq 0$.
2. If a function has a constant derivative then it is linear, and conversely.
3. If $xy = 0$ then $x = 0$ or $y = 0$, and conversely.
4. If $a \in \mathbb{Q}$ then $5a \in \mathbb{Q}$, and if $5a \in \mathbb{Q}$ then $a \in \mathbb{Q}$.
5. For an occurrence to become an adventure, it is necessary and sufficient for one to recount it. (Jean-Paul Sartre)

3.5 Truth Tables for Statements

You should now know the truth tables for \wedge , \vee , \neg , \Rightarrow and \Leftrightarrow . They should be *internalized* as well as memorized. You must understand the symbols thoroughly, for we now combine them to form more complex statements.

For example, suppose we want to convey that one or the other of P and Q is true but they are not both true. No single symbol expresses this, but we could

combine them as

$$(P \vee Q) \wedge \neg(P \wedge Q),$$

which literally means:

P or Q is true, and it is not the case that both P and Q are true.

This statement will be true or false depending on the truth values of P and Q . In fact we can make a truth table for the entire statement. Begin as usual by listing the possible true/false combinations of P and Q on four lines. The statement $(P \vee Q) \wedge \neg(P \wedge Q)$ contains the individual statements $(P \vee Q)$ and $(P \wedge Q)$, so we next tally their truth values in the third and fourth columns. The fifth column lists values for $\neg(P \wedge Q)$, and these are just the opposites of the corresponding entries in the fourth column. Finally, combining the third and fifth columns with \wedge , we get the values for $(P \vee Q) \wedge \neg(P \wedge Q)$ in the sixth column.

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge \neg(P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

This truth table tells us that $(P \vee Q) \wedge \neg(P \wedge Q)$ is true precisely when one but not both of P and Q are true, so it has the meaning we intended. (Notice that the middle three columns of our truth table are just “helper columns” and are not necessary parts of the table. In writing truth tables, you may choose to omit such columns if you are confident about your work.)

For another example, take this familiar statement about two numbers x and y :

The product xy equals zero if and only if $x = 0$ or $y = 0$.

This can be modeled as $(xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$. If we introduce letters P, Q and R for the statements $xy = 0$, $x = 0$ and $y = 0$, it becomes $P \Leftrightarrow (Q \vee R)$. Notice that the parentheses are necessary here, for without them we wouldn't know whether to read the statement as $P \Leftrightarrow (Q \vee R)$ or $(P \Leftrightarrow Q) \vee R$.

Making a truth table for $P \Leftrightarrow (Q \vee R)$ entails a line for each T/F combination for the three statements P, Q and R . The eight combinations are tallied below.

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

We fill in the fourth column using our knowledge of the truth table for \vee . Finally the fifth column is filled in by combining the first and fourth columns with our understanding of the truth table for \Leftrightarrow . The resulting table gives the true/false values of $P \Leftrightarrow (Q \vee R)$ for all values of P, Q and R .

Notice that when we plug in various values for x and y , the statements $P : xy = 0$, $Q : x = 0$ and $R : y = 0$ have various truth values, but the statement $P \Leftrightarrow (Q \vee R)$ is always true. For example, if $x = 2$ and $y = 3$, then P, Q and R are all false. This scenario is described in the last row of the table, and there we see that $P \Leftrightarrow (Q \vee R)$ is true. Likewise if $x = 0$ and $y = 7$, then P and Q are true and R is false, a scenario described in the second line of the table, where again $P \Leftrightarrow (Q \vee R)$ is true. There is a simple reason why $P \Leftrightarrow (Q \vee R)$ is true for any values of x and y : It is that $P \Leftrightarrow (Q \vee R)$ represents $(xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$, which is a *true mathematical statement*. It is absolutely impossible for it to be false.

This may make you wonder about the lines in the table where $P \Leftrightarrow (Q \vee R)$ is false. Why are they there? The reason is that $P \Leftrightarrow (Q \vee R)$ can also represent a false statement. To see how, imagine that at the end of the semester your professor makes this promise:

You pass the class if and only if you get an “A” on the final or you get a “B” on the final.

This promise has the form $P \Leftrightarrow (Q \vee R)$, so its truth values are tabulated in the above table. Imagine it turned out that you got an “A” on the exam but failed the course. Then surely your professor lied to you. In fact, P is false, Q is true and R is false. This scenario is reflected in the sixth line of the table, and indeed $P \Leftrightarrow (Q \vee R)$ is false (i.e., it is a lie).

The moral of this example is that people can lie, but true mathematical statements *never* lie.

We close this section with a word about the use of parentheses. The symbol \neg is analogous to the minus sign in algebra. It negates the expression it precedes. Thus $\neg P \vee Q$ means $(\neg P) \vee Q$, not $\neg(P \vee Q)$. In $\neg(P \vee Q)$, the value of the entire expression $P \vee Q$ is negated.

Exercises for Section 3.5

Write a truth table for the logical statements in problems 1–9:

1. $P \vee (Q \Rightarrow R)$
2. $(Q \vee R) \Leftrightarrow (R \wedge Q)$
3. $\neg(P \Rightarrow Q)$
4. $\neg(P \vee Q) \vee (\neg P)$
5. $(P \wedge \neg P) \vee Q$
6. $(P \wedge \neg P) \wedge Q$
7. $(P \wedge \neg P) \Rightarrow Q$
8. $P \vee (Q \wedge \neg R)$
9. $\neg(\neg P \vee \neg Q)$

10. Suppose the statement $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is false. Find the truth values of P, Q, R and S . (This can be done without a truth table.)
11. Suppose P is false and that the statement $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true. Find the truth values of R and S . (This can be done without a truth table.)

3.6 Logical Equivalence

In contemplating the truth table for $P \Leftrightarrow Q$, you probably noticed that $P \Leftrightarrow Q$ is true exactly when P and Q are both true or both false. In other words, $P \Leftrightarrow Q$ is true precisely when at least one of the statements $P \wedge Q$ or $\neg P \wedge \neg Q$ is true. This may tempt us to say that $P \Leftrightarrow Q$ means the same thing as $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

To see if this is really so, we can write truth tables for $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\neg P \wedge \neg Q)$. In doing this, it is more efficient to put these two statements into the same table, as follows. (This table has helper columns for the intermediate expressions $\neg P$, $\neg Q$, $(P \wedge Q)$ and $(\neg P \wedge \neg Q)$.)

P	Q	$\neg P$	$\neg Q$	$(P \wedge Q)$	$(\neg P \wedge \neg Q)$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	$P \Leftrightarrow Q$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T

The table shows that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ have the same truth value, no matter the values P and Q . It is as if $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ are algebraic expressions that are equal no matter what is “plugged into” variables P and Q . We express this state of affairs by writing

$$P \Leftrightarrow Q = (P \wedge Q) \vee (\neg P \wedge \neg Q)$$

and saying that $P \Leftrightarrow Q$ and $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ are **logically equivalent**.

In general, two statements are **logically equivalent** if their truth values match up line-for-line in a truth table.

Logical equivalence is important because it can give us different (and potentially useful) ways of looking at the same thing. As an example, the following table shows that $P \Rightarrow Q$ is logically equivalent to $(\neg Q) \Rightarrow (\neg P)$.

P	Q	$\neg P$	$\neg Q$	$(\neg Q) \Rightarrow (\neg P)$	$P \Rightarrow Q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The fact that $P \Rightarrow Q = (\neg Q) \Rightarrow (\neg P)$ is useful because so many theorems have the form $P \Rightarrow Q$. As we will see in Chapter 5, proving such a theorem may be easier if we express it in the logically equivalent form $(\neg Q) \Rightarrow (\neg P)$.

There are two pairs of logically equivalent statements that come up again and again throughout this book and beyond. They are prevalent enough to be dignified by a special name: **DeMorgan’s laws**.

Fact 3.1. (DeMorgan's Laws)

- (1) $\neg(P \wedge Q) = (\neg P) \vee (\neg Q)$
- (2) $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$

The first of DeMorgan's laws is verified by the following table. You are asked to verify the second in one of the exercises.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

DeMorgan's laws are actually very natural and intuitive. Consider the statement $\neg(P \wedge Q)$, which we can interpret as meaning that *it is not the case that both P and Q are true*. If it is not the case that both P and Q are true, then at least one of P or Q is false, in which case $(\neg P) \vee (\neg Q)$ is true. Thus $\neg(P \wedge Q)$ means the same thing as $(\neg P) \vee (\neg Q)$.

DeMorgan's laws can be very useful. Suppose we happen to know that some statement having form $\neg(P \vee Q)$ is true. The second of DeMorgan's laws tells us that $(\neg Q) \wedge (\neg P)$ is also true, hence $\neg P$ and $\neg Q$ are both true as well. Being able to quickly obtain such additional pieces of information can be extremely useful.

Here is a summary of some significant logical equivalences. Those that are not immediately obvious can be verified with a truth table.

$$P \Rightarrow Q = (\neg Q) \Rightarrow (\neg P) \quad \text{Contrapositive law} \quad (3.1)$$

$$\left. \begin{aligned} \neg(P \wedge Q) &= \neg P \vee \neg Q \\ \neg(P \vee Q) &= \neg P \wedge \neg Q \end{aligned} \right\} \text{DeMorgan's laws} \quad (3.2)$$

$$\left. \begin{aligned} P \wedge Q &= Q \wedge P \\ P \vee Q &= Q \vee P \end{aligned} \right\} \text{Commutative laws} \quad (3.3)$$

$$\left. \begin{aligned} P \wedge (Q \vee R) &= (P \wedge Q) \vee (P \wedge R) \\ P \vee (Q \wedge R) &= (P \vee Q) \wedge (P \vee R) \end{aligned} \right\} \text{Distributive laws} \quad (3.4)$$

$$\left. \begin{aligned} P \wedge (Q \wedge R) &= (P \wedge Q) \wedge R \\ P \vee (Q \vee R) &= (P \vee Q) \vee R \end{aligned} \right\} \text{Associative laws} \quad (3.5)$$

Notice how the distributive law $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$ has the same structure as the distributive law $p \cdot (q + r) = p \cdot q + p \cdot r$ from algebra. Concerning the associative laws, the fact that $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$ means that the position of the parentheses is irrelevant, and we can write this as $P \wedge Q \wedge R$ without ambiguity. Similarly, we may drop the parentheses in an expression such as $P \vee (Q \vee R)$.

But parentheses are essential when there is a mix of \wedge and \vee , as in $P \vee (Q \wedge R)$. Indeed, $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge R$ are **not** logically equivalent. (See Exercise 13 for Section 3.6, below.)

Exercises for Section 3.6

A. Use truth tables to show that the following statements are logically equivalent.

1. $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
2. $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$
3. $P \Rightarrow Q = (\neg P) \vee Q$
4. $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$
5. $\neg(P \vee Q \vee R) = (\neg P) \wedge (\neg Q) \wedge (\neg R)$
6. $\neg(P \wedge Q \wedge R) = (\neg P) \vee (\neg Q) \vee (\neg R)$
7. $P \Rightarrow Q = (P \wedge \neg Q) \Rightarrow (Q \wedge \neg Q)$
8. $\neg P \Leftrightarrow Q = (P \Rightarrow \neg Q) \wedge (\neg Q \Rightarrow P)$

B. Decide whether or not the following pairs of statements are logically equivalent.

9. $P \wedge Q$ and $\neg(\neg P \vee \neg Q)$
10. $(P \Rightarrow Q) \vee R$ and $\neg((P \wedge \neg Q) \wedge \neg R)$
11. $(\neg P) \wedge (P \Rightarrow Q)$ and $\neg(Q \Rightarrow P)$
12. $\neg(P \Rightarrow Q)$ and $P \wedge \neg Q$
13. $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge R$
14. $P \wedge (Q \vee \neg Q)$ and $(\neg P) \Rightarrow (Q \wedge \neg Q)$

Solutions for Chapter 3

Section 3.1

Decide whether or not the following are statements. In the case of a statement, say if it is true or false.

1. Every real number is an even integer. (Statement, False)
3. If x and y are real numbers and $5x = 5y$, then $x = y$. (Statement, True)
5. Sets \mathbb{Z} and \mathbb{N} are infinite. (Statement, True)
7. The derivative of any polynomial of degree 5 is a polynomial of degree 6. (Statement, False)
9. $\cos(x) = -1$
This is not a statement. It is an open sentence because whether it's true or false depends on the value of x .
11. The integer x is a multiple of 7.
This is an open sentence, and not a statement.
13. Either x is a multiple of 7, or it is not.
This is a statement, for the sentence is true no matter what x is.
15. In the beginning God created the heaven and the earth.
This is a statement, for it is either definitely true or definitely false. There is some controversy over whether it's true or false, but no one claims that it is neither true nor false.

Section 3.2

Express each statement as one of the forms $P \wedge Q$, $P \vee Q$, or $\neg P$. Be sure to also state exactly what statements P and Q stand for.

1. The number 8 is both even and a power of 2.
 $P \wedge Q$
 P : 8 is even
 Q : 8 is a power of 2
Note: Do not say " Q : a power of 2," because that is not a statement.
3. $x \neq y$ $\neg(x = y)$ (Also $\neg P$ where $P : x = y$.)
5. $y \geq x$ $\neg(y < x)$ (Also $\neg P$ where $P : y < x$.)
7. The number x equals zero, but the number y does not.
 $P \wedge \neg Q$
 $P : x = 0$
 $Q : y = 0$
9. $x \in A - B$
 $(x \in A) \wedge \neg(x \in B)$
11. $A \in \{X \in \mathcal{P}(\mathbb{N}) : |\overline{X}| < \infty\}$
 $(A \subseteq \mathbb{N}) \wedge (|\overline{A}| < \infty)$.
13. Human beings want to be good, but not too good, and not all the time.
 $P \wedge \neg Q \wedge \neg R$
 P : Human beings want to be good.
 Q : Human beings want to be too good.
 R : Human beings want to be good all the time.

Section 3.3

Without changing their meanings, convert each of the following sentences into a sentence having the form “If P , then Q .”

1. A matrix is invertible provided that its determinant is not zero.
Answer: If a matrix has a determinant not equal to zero, then it is invertible.
3. For a function to be integrable, it is necessary that it is continuous.
Answer: If a function is integrable, then it is continuous.
5. An integer is divisible by 8 only if it is divisible by 4.
Answer: If an integer is divisible by 8, then it is divisible by 4.
7. A series converges whenever it converges absolutely.
Answer: If a series converges absolutely, then it converges.
9. A function is integrable provided the function is continuous.
Answer: If a function is continuous, then that function is integrable.
11. You fail only if you stop writing.
Answer: If you fail, then you have stopped writing.
13. Whenever people agree with me I feel I must be wrong.
Answer: If people agree with me, then I feel I must be wrong.

Section 3.4

Without changing their meanings, convert each of the following sentences into a sentence having the form “ P if and only if Q .”

1. For a matrix to be invertible, it is necessary and sufficient that its determinant is not zero.
Answer: A matrix is invertible if and only if its determinant is not zero.
3. If $xy = 0$ then $x = 0$ or $y = 0$, and conversely.
Answer: $xy = 0$ if and only if $x = 0$ or $y = 0$
5. For an occurrence to become an adventure, it is necessary and sufficient for one to recount it.
Answer: An occurrence becomes an adventure if and only if one recounts it.

Section 3.5

1. Write a truth table for $P \vee (Q \Rightarrow R)$
3. Write a truth table for $\neg(P \Rightarrow Q)$

P	Q	R	$Q \Rightarrow R$	$P \vee (Q \Rightarrow R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	T	T

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

5. Write a truth table for $(P \wedge \neg P) \vee Q$
7. Write a truth table for $(P \wedge \neg P) \Rightarrow Q$

P	Q	$(P \wedge \neg P)$	$(P \wedge \neg P) \vee Q$
T	T	F	T
T	F	F	F
F	T	F	T
F	F	F	F

P	Q	$(P \wedge \neg P)$	$(P \wedge \neg P) \Rightarrow Q$
T	T	F	T
T	F	F	T
F	T	F	T
F	F	F	T

9. Write a truth table for $\neg(\neg P \vee \neg Q)$.

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(\neg P \vee \neg Q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

11. Suppose P is false and that the statement $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true. Find the truth values of R and S . (This can be done without a truth table.)

Answer: Since P is false, it follows that $(P \wedge Q)$ is false also. But then in order for $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ to be true, it must be that $(R \Rightarrow S)$ is false. The only way for $(R \Rightarrow S)$ to be false is if R is true and S is false.

Section 3.6

- A. Use truth tables to show that the following statements are logically equivalent.

1. $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

P	Q	R	$Q \vee R$	$P \wedge Q$	$P \wedge R$	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Thus since their columns agree, the two statements are logically equivalent.

3. $P \Rightarrow Q = (\neg P) \vee Q$

P	Q	$\neg P$	$(\neg P) \vee Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Thus since their columns agree, the two statements are logically equivalent.

5. $\neg(P \vee Q \vee R) = (\neg P) \wedge (\neg Q) \wedge (\neg R)$

P	Q	R	$P \vee Q \vee R$	$\neg P$	$\neg Q$	$\neg R$	$\neg(P \vee Q \vee R)$	$(\neg P) \wedge (\neg Q) \wedge (\neg R)$
T	T	T	T	F	F	F	F	F
T	T	F	T	F	F	T	F	F
T	F	T	T	F	T	F	F	F
T	F	F	T	F	T	T	F	F
F	T	T	T	T	F	F	F	F
F	T	F	T	T	F	T	F	F
F	F	T	T	T	T	F	F	F
F	F	F	F	T	T	T	T	T

Thus since their columns agree, the two statements are logically equivalent.

7. $P \Rightarrow Q = (P \wedge \neg Q) \Rightarrow (Q \wedge \neg Q)$

P	Q	$\neg Q$	$P \wedge \neg Q$	$Q \wedge \neg Q$	$(P \wedge \neg Q) \Rightarrow (Q \wedge \neg Q)$	$P \Rightarrow Q$
T	T	F	F	F	T	T
T	F	T	T	F	F	F
F	T	F	F	F	T	T
F	F	T	F	F	T	T

Thus since their columns agree, the two statements are logically equivalent.

B. Decide whether or not the following pairs of statements are logically equivalent.

9. By DeMorgan's law, we have $\neg(\neg P \vee \neg Q) = \neg\neg P \wedge \neg\neg Q = P \wedge Q$. Thus the two statements are logically equivalent.

11. $(\neg P) \wedge (P \Rightarrow Q)$ and $\neg(Q \Rightarrow P)$

P	Q	$\neg P$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(\neg P) \wedge (P \Rightarrow Q)$	$\neg(Q \Rightarrow P)$
T	T	F	T	T	F	F
T	F	F	F	T	F	F
F	T	T	T	F	T	T
F	F	T	T	T	T	F

The columns for the two statements do not quite agree, thus the two statements are **not logically equivalent**.

13. $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge R$ are **not logically equivalent** because if $P = T$ and $Q = R = F$, then the first statement is true and the second is false.