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Midterm Exam

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MATH 200

October 26, 2022

1. (35 pts.) Evaluate the following limits. Show steps, as appropriate.

(a) $\lim_{x\to 0} \frac{\sin(x)}{x} = \boxed{1}$ (standard fact)

(b) $\lim_{x\to\infty} \tan^{-1}\left(1+\frac{1}{x}\right) = \tan^{-1}\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\right) = \tan^{-1}(1+0) = \tan^{-1}(1) = \boxed{\frac{\pi}{4}}$

(c) $\lim_{x\to\infty} e^{1/x} = e^{\lim_{x\to\infty}\frac{1}{x}} = e^{0} = \boxed{1}$

(d) $\lim_{x\to 1} \frac{x^2-4x+3}{x^2+4x-5} = \lim_{x\to 1}\frac{\cancel{(x-1)}(x-3)}{\cancel{(x-1)}(x+5)} = \lim_{x\to 1}\frac{x-3}{x+5} = \frac{1-3}{1+5} = \frac{-2}{6} = \boxed{-\frac{1}{3}}$

$\frac{0}{0}$

(e) $\lim_{h\to 0} \frac{\frac{1}{2+h}-\frac{1}{2}}{h} = \lim_{h\to 0}\frac{\frac{1}{2+h}-\frac{1}{2}}{h}\cdot\frac{2(2+h)}{2(2+h)} = \lim_{h\to 0}\frac{2-(2+h)}{h\cdot 2(2+h)}$

$\frac{0}{0}$

$= \lim_{h\to 0}\frac{-\cancel{h}}{\cancel{h}\cdot 2(2+h)} = \lim_{h\to 0}\frac{-1}{2(2+h)} = \frac{-1}{2(2+0)} = \boxed{-\frac{1}{4}}$

(f) $\lim_{x\to 4^+} \frac{(-x+4)(x+2)}{|-x+4|} = \lim_{x\to 4^+}\frac{\cancel{(-x+4)}(x+2)}{-\cancel{(-x+4)}} = \lim_{x\to 4^+} -(x+2)$

$= -(4+2) = \boxed{-6}$

4 x

Note $-x+4$ is negative, so $|-x+4| = -(-x+4)$

approaches $(-4+5)(4+2) = 6$

(g) $\lim_{x\to 4^+} \frac{(-x+5)(x+2)}{|-x+4|} = \boxed{\infty}$

approaches 0, positive

2. (5 pts.) Use a limit definition of a derivative to find the derivative of $f(x) = 2 - 3x^2$.

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{(2 - 3z^2) - (2 - 3x^2)}{z - x}$$

$$= \lim_{z \to x} \frac{2 - 3z^2 - 2 + 3x^2}{z - x}$$

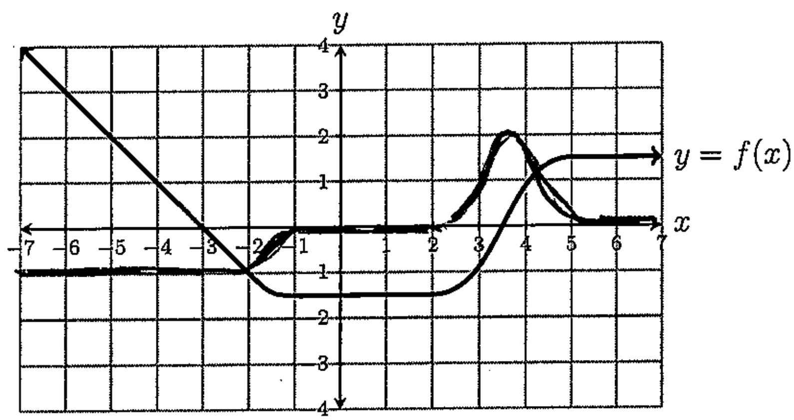
$$= \lim_{z \to x} \frac{-3z^2 + 3x^2}{z - x} = \lim_{z \to x} \frac{-3(z^2 - x^2)}{z - x}$$

$$= \lim_{z \to x} \frac{-3(z - x)(z + x)}{z - x}$$

$$= \lim_{z \to x} -3(z + x) = -3(x + x) = -3(2x) = \boxed{-6x}$$

3. (5 pts.) The graph of a function $f(x)$ is shown. Using the same grid, sketch the graph of $f'(x)$.

$y = f'(x)$



4. (5 pts.) Find all points (x, y) on the graph of $y = x + \dfrac{1}{x - 3}$ where the tangent line is horizontal.

Solve $y' = 0$

$$1 - \frac{1}{(x-3)^2} = 0$$

$$(x-3)^2\left(1 - \frac{1}{(x-3)^2}\right) = 0(x-3)^2$$

$$(x-3)^2 - 1 = 0$$

$$x^2 - 6x + 9 - 1 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x-2)(x-4) = 0$$

$\boxed{x = 2}$ $\boxed{x = 4}$

Points:

$$\left(2, 2 + \frac{1}{2-3}\right) = (2, 2-1) = \boxed{(2, 1)}$$

$$\left(4, 4 + \frac{1}{4-3}\right) = (4, 4+1) = \boxed{(4, 5)}$$

5. (30 pts.) Find the indicated derivatives.

(a) $f(\theta) = 5 + \ln(\pi\theta) + \sqrt{\theta^3} = 5 + \ln(\pi\theta) + \theta^{3/2}$

$f'(\theta) = 0 + \frac{\pi}{\pi\theta} + \frac{3}{2}\theta^{1/2} = \frac{1}{\theta} + \frac{3}{2}\theta^{1/2} = \boxed{\frac{1}{\theta} + \frac{3}{2}\sqrt{\theta}}$

$f''(\theta) = -\frac{1}{\theta^2} + \frac{3}{2}\cdot\frac{1}{2}\theta^{-1/2} = \boxed{\frac{3}{4\sqrt{\theta}} - \frac{1}{\theta^2}}$

(b) $D_x\left[\frac{x}{x^3+x^2+1}\right] = \frac{(1)(x^3+x^2+1) - x(3x^2+2x)}{(x^3+x^2+1)^2}$

$= \boxed{\frac{1 - 2x^3 - x^2}{(x^3+x^2+1)^2}}$

(c) $D_x\left[e^{4x}\sqrt{3x+1}\right] = 4e^{4x}\sqrt{3x+1} + e^{4x} D_x\left[(3x+1)^{\frac{1}{2}}\right]$

$= 4e^{4x}\sqrt{3x+1} + e^{4x}\frac{1}{2}(3x+1)^{-\frac{1}{2}}\cdot 3 = \boxed{e^{4x}\left(4\sqrt{3x+1} + \frac{3}{2\sqrt{3x+1}}\right)}$

(d) $D_x\left[\ln\left(\sec(x^3)\right)\right] = \frac{1}{\sec(x^3)} D_x\left[\sec(x^3)\right] = \frac{1}{\sec(x^3)}\sec(x^3)\tan(x^3)3x^2$

$= \boxed{\tan(x^3)\,3x^2}$

(e) $D_x\left[\tan^{-1}(\pi x)\right] = \frac{1}{1+(\pi x)^2}\pi = \boxed{\frac{\pi}{1+\pi^2x^2}}$

6. (5 pts.) Consider the equation $x\sin(y) = y^3$. Use implicit differentiation to find $\frac{dy}{dx}$.

$$D_x\left[x\sin(y)\right] = D_x\left[y^3\right] \qquad \{y = f(x)\}$$

$$1\cdot\sin(y) + x\cos(y)y' = 3y^2y'$$

$$\sin(y) = 3y^2y' - x\cos(y)y'$$

$$\sin(y) = y'(3y^2 - x\cos(y))$$

$$\boxed{\frac{\sin(y)}{3y^2 - x\cos(y)} = y'}$$

7. (5 pts.) Use logarithmic differentiation to find the derivative of $f(x) = (1+2x)^x$.

$$\ln|f(x)| = \ln\left|(1+2x)^x\right|$$

$$\ln|f(x)| = x\ln|1+2x|$$

$$D_x\left[\ln|f(x)|\right] = D_x\left[x\ln|1+2x|\right]$$

$$\frac{f'(x)}{f(x)} = 1\ln|1+2x| + x\frac{2}{1+2x}$$

$$f'(x) = f(x)\left(\ln|1+2x| + \frac{2x}{1+2x}\right)$$

$$\boxed{f'(x) = (1+2x)^x\left(\ln|1+2x| + \frac{2x}{1+2x}\right)}$$

8. (10 pts.) A rock is thrown from a tower at time $t = 0$. At time t (in seconds) it has a height of $s(t) = 48 + 32t - 16t^2$ feet. Please show your work in answering the following questions.

(a) When does the rock hit the ground?

$s(t) = $ (height at time t)

Solve $s(t) = 0$

$$48 + 32t - 16t^2 = 0$$
$$-16(t^2 - 2t - 3) = 0$$
$$-16(t+1)(t-3)$$
$$t = -1 \qquad \boxed{t = 3}$$

Ignore negative time

$\boxed{\text{Rock hits ground at time } = 3 \text{ seconds}}$

(b) What is its velocity when it hits the ground?

$$v(t) = s'(t) = 32 - 32t$$
$$v(3) = 32 - 32 \cdot 3 = -32 \cdot 2 = \boxed{-64 \text{ ft/sec}}$$

9. (**Bonus:** 5 pts.) Sand falls at a rate of 6 cubic feet per minute, making a conical pile whose height h is always half its radius r. Find the rate of change of the radius r (in feet/min) when $r = 2$ feet.

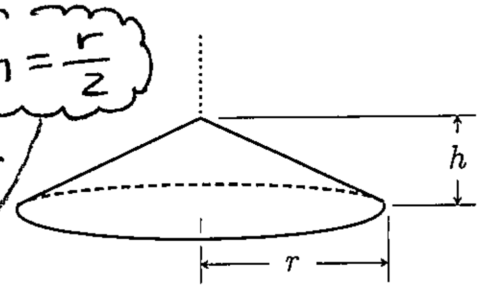
V = volume

r = radius

Know $\frac{dV}{dt} = 6$

Want $\frac{dr}{dt}$ when $r = 2$

$h = \frac{r}{2}$



$$V = \frac{1}{3}\pi r^2 \frac{r}{2}$$

$$V = \frac{\pi}{6} r^3$$

$$D_t[V] = D_t\left[\frac{\pi}{6} r^3\right]$$

$$\frac{dV}{dt} = \frac{\pi}{6} 3r^2 \frac{dr}{dt}$$

$$6 = \frac{\pi}{2} r^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{6}{\frac{\pi r^2}{2}} = \frac{12}{\pi r^2}$$

Ans $\left.\frac{dr}{dt}\right|_{r=2} = \frac{12}{\pi \cdot 2^2}$

$$= \boxed{\frac{3}{\pi} \text{ ft/min}}$$

Geometry formula: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.