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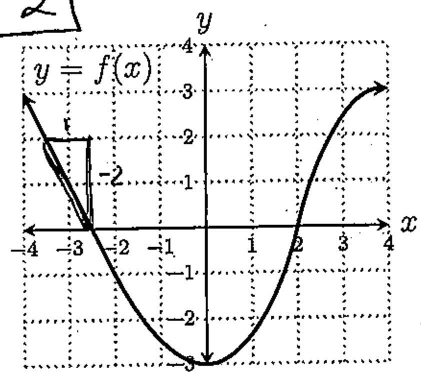
Directions: Closed book, closed notes, no calculators. Put all phones, etc., away. You will need only a pencil or pen.

1. (10 points) Answer the questions about the function f graphed below.

(a) $\lim\limits_{x\to\infty} f\left(\frac{1}{x}\right) =$ $f\left(\lim\limits_{x\to\infty} \frac{1}{x}\right) = f(0) = \boxed{-3}$

(b) $\lim\limits_{h\to 0} \frac{f(-3+h)-f(-3)}{h} =$ $f'(-3) = \frac{\text{rise}}{\text{run}} = \frac{-2}{1} = \boxed{-2}$



(c) $\lim\limits_{x\to -2} \frac{f(x)}{(1+f(x))^2} =$ $\boxed{-\infty}$

approaching -1

approaching 0, positive

(d) $\lim\limits_{x\to 2} \frac{\sin(f(x))+1}{f(x)+1} =$ $\frac{\sin(0)+1}{0+1} = \frac{1}{1} = \boxed{1}$

(e) $\lim\limits_{x\to 2} \frac{\sin(f(x))}{f(x)} =$ $\boxed{1}$

2. (20 points) Find the limits

(a) $\lim\limits_{x\to 0^+} \sin^{-1}(x-1) =$ $\sin^{-1}(0-1) = \sin^{-1}(-1) = \boxed{-\frac{\pi}{2}}$

(b) $\lim\limits_{x\to e} 5\ln(x^3) =$ $5 \lim\limits_{x\to e} \ln(x^3) = 5\ln\left(\lim\limits_{x\to e} x^3\right) = 5\ln(e^3) = 5\cdot 3 = \boxed{15}$

(c) $\lim\limits_{x\to 3} \frac{x-3}{x^2-7x+12} =$ $\lim\limits_{x\to 3} \frac{\cancel{x-3}}{\cancel{(x-3)}(x-4)} = \lim\limits_{x\to 3} \frac{1}{x-4} = \frac{1}{3-4} = \boxed{-1}$

(d) $\lim\limits_{x\to 1} \frac{\frac{1}{x}-1}{x-1} =$ $\lim\limits_{x\to 1} \frac{\frac{1}{x}-1}{x-1}\cdot\frac{x}{x} = \lim\limits_{x\to 1} \frac{1-x}{(x-1)x} = \lim\limits_{x\to 1} \frac{-(x-1)}{(x-1)x}$

$= \lim\limits_{x\to 1} \frac{-1}{x} = \frac{-1}{1} = \boxed{-1}$

3. (7 points) Use a **limit definition** of the derivative to find the derivative of $f(x) = \sqrt{1-x}$.

$$f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x} = \lim_{z\to x} \frac{\sqrt{1-z}-\sqrt{1-x}}{z-x}$$

$$= \lim_{z\to x} \frac{\sqrt{1-z}-\sqrt{1-x}}{z-x} \cdot \frac{\sqrt{1-z}+\sqrt{1-x}}{\sqrt{1-z}+\sqrt{1-x}}$$

$$= \lim_{z\to x} \frac{1-z-(1-x)}{(z-x)(\sqrt{1-z}+\sqrt{1-x})} = \lim_{z\to x} \frac{-z+x}{(z-x)(\sqrt{1-z}+\sqrt{1-x})}$$

$$= \lim_{z\to x} \frac{-\cancel{(z-x)}}{\cancel{(z-x)}(\sqrt{1-z}+\sqrt{1-z})} = \lim_{z\to x} \frac{-1}{\sqrt{1-z}+\sqrt{1-x}} = \frac{-1}{\sqrt{1-x}+\sqrt{1-x}}$$

$$= \boxed{\frac{-1}{2\sqrt{1-x}}}$$

Therefore $\boxed{f'(x) = \frac{-1}{2\sqrt{1-x}}}$

4. (7 points) An object moving on a straight line is $s(t) = t^3 - 3t^2$ feet from its starting point at time t seconds. Find its velocity when its acceleration is 12 feet per second per second.

velocity: $v(t) = s'(t) = 3t^2 - 6t$

Acceleration: $a(t) = v'(t) = 6t - 6$

To find when acceleration is 12, solve $a(t) = 12 \Rightarrow$

$6t - 6 = 12 \Rightarrow 6t = 18 \Rightarrow \boxed{t = 3 \text{ seconds}}$

Thus acceleration is 12 at $t = 3$ seconds.

At this time the velocity is $v(3) = 3\cdot 3^2 - 6\cdot 3 = \boxed{9 \text{ ft/sec}}$

5. (7 points) Suppose $f(x) = x^2 + 2x^3$ and $g(x) = x^2 - 2x^3 + 48x$. Find all x for which the tangent to $y = f(x)$ at $(x, f(x))$ is parallel to the tangent to $y = g(x)$ at $(x, g(x))$.

Solve $f'(x) = g'(x)$

$2x + 6x^2 = 2x - 6x^2 + 48$

$12x^2 - 48 = 0$

$12(x^2 - 4) = 0$

$12(x-2)(x+2) = 0$

$x = 2 \qquad x = -2$

Answer: Tangents are parallel when $x = -2$ and $x = 2$

6. (35 points) Find the derivatives of these functions. You do **not** need to simplify your answers.

(a) $f(x) = \frac{\sqrt{2}}{x} + \pi x$ $= \sqrt{2}x^{-1} + \pi x.$ $f'(x) = -\sqrt{2}x^{-2} + \pi = \boxed{\pi - \frac{\sqrt{2}}{x^2}}$

(b) $f(x) = \cos(x)\sin(x)$ $f'(x) = -\sin(x)\sin(x) + \cos(x)\cos(x)$

$= \boxed{\cos^2(x) - \sin^2(x)}$

(c) $f(x) = \cos(\sin(x))$ $f'(x) = \boxed{-\sin(\sin(x)) \cdot \cos(x)}$

(d) $f(x) = \tan^{-1}(-x)$ $f'(x) = \frac{1}{1+(-x)^2}(-1) = \boxed{\frac{-1}{1+x^2}}$

(e) $f(x) = \ln\left(e^{x^2-3x} + x\right)$ $f'(x) = \boxed{\frac{e^{x^2-3x}(2x-3)+1}{e^{x^2-3x}+x}}$

(f) $f(x) = \frac{1}{x^2+5x-7}$ $= (x^2+5x-7)^{-1}$ $f'(x) = -(x^2+5x-7)^{-2}(2x+5)$

$= \boxed{\frac{-2x-5}{(x^2+5x-7)^2}}$

(g) $f(x) = \sqrt{\frac{x+1}{x-1}}^3$ $= \left(\frac{x+1}{x-1}\right)^{\frac{3}{2}}$

$f'(x) = \frac{3}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} \frac{1(x-1)-(x+1)(1)}{(x-1)^2} = \frac{3}{2}\sqrt{\frac{x+1}{x-1}} \frac{-2}{(x-1)^2}$

$= \boxed{-3\sqrt{\frac{x+1}{x-1}} \cdot \frac{1}{(x-1)^2}}$

7. (7 points) Given the equation $\frac{x}{y} = y^5 + x$, find y'.

$$D_x\left[\frac{x}{y}\right] = D_x\left[y^5 + x\right]$$

$$\frac{y - xy'}{y^2} = 5y^4y' + 1$$

$$y - xy' = y^2(5y^4y' + 1)$$

$$y - xy' = 5y^6y' + y^2$$

$$y - y^2 = 5y^6y' + xy'$$

$$y - y^2 = (5y^6 + x)y'$$

$$\boxed{y' = \frac{y - y^2}{5x^6 + x}}$$

8. (7 points) Find the derivative of $f(x) = x^{\ln(x)}$.

Logarithmic Differentiation.

$$y = x^{\ln(x)}$$

$$\ln(y) = \ln\left(x^{\ln(x)}\right)$$

$$\ln(y) = \ln(x) \cdot \ln(x)$$

$$D_x\left[\ln(y)\right] = D_x\left[\ln(x) \cdot \ln(x)\right]$$

$$\frac{y'}{y} = \frac{1}{x}\ln(x) + \ln(x)\frac{1}{x}$$

$$\frac{y'}{y} = 2\,\frac{\ln(x)}{x}$$

$$y' = 2y\,\frac{\ln(x)}{x}$$

$$\boxed{y' = \frac{2x^{\ln(x)}\ln(x)}{x}}$$