Name: ____________________

Math 200 Midterm Exam

March 17, 2022

Directions: Closed book, closed notes, no calculators. Put all phones, etc., away. You will need only a pencil or pen.

1. (10 points) Draw the graph of one function $f(x)$ meeting **all** of the following conditions.

   (a) $\lim_{z\to 3} f(x) = \infty$

   (b) $\lim_{z\to \infty} f(x) = \infty$

   (c) $\lim_{z\to -\infty} f(x) = 2$

   (d) f is continuous on $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$.

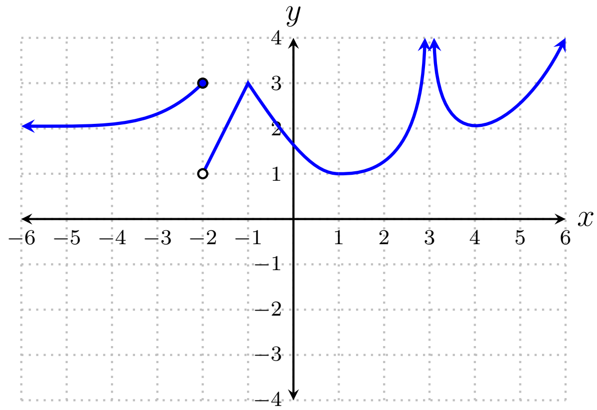
   (e) $f(1) = 1$

   (f) $f'(1) = 0$

   (g) $f'(-1)$ does not exist

   (h) $\lim_{z\to -2^+} f(x) = 1$

   (i) $\lim_{z\to -2^-} f(x) = 3$



2. (24 points) Find the limits.

   (a) $\lim_{x\to\infty} \tan^{-1}(x) = \boxed{\dfrac{\pi}{2}}$

   (b) $\lim_{x\to 1/2} \sin^{-1}(x) = \sin^{-1}(1/2) = \left(\begin{array}{l}\text{angle } \theta \text{ for which} \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and} \\ \sin(\theta) = 1/2\end{array}\right) = \boxed{\dfrac{\pi}{6}}$

   (c) $\lim_{z\to 0} \dfrac{e^z - e^0}{z-0} = e^0 = \boxed{1}$

   Because if $f(x) = e^x$, then $f'(x) = \lim_{z\to x} \dfrac{e^z - e^x}{z - x} = e^x$, and therefore $\lim_{z\to 0} \dfrac{e^z - e^0}{z-0} = e^0$.

   (d) $\lim_{x\to 2} \dfrac{\frac{4}{x} - 1}{x-4} = \dfrac{\frac{4}{2} - 1}{2-4} = \dfrac{2-1}{2-4} = \boxed{-\dfrac{1}{2}}$

   (e) $\lim_{x\to 4} \dfrac{\frac{4}{x} - 1}{x-4} = \lim_{x\to 4} \dfrac{\frac{4}{x} - 1}{x-4} \cdot \dfrac{x}{x} = \lim_{x\to 4} \dfrac{4-x}{(x-4)x} = \lim_{x\to 4} \dfrac{-1}{x} = \boxed{-\dfrac{1}{4}}$

   (f) $\lim_{x\to \infty} \dfrac{\frac{4}{x} - 1}{x-4} = \lim_{x\to \infty} \dfrac{\frac{4}{x} - 1}{x-4} \cdot \dfrac{x}{x} = \lim_{x\to \infty} \dfrac{4-x}{(x-4)x} = \lim_{x\to \infty} \dfrac{-1}{x} = \boxed{0}$

3. (6 points) Use a **limit definition** of the derivative to find the derivative of $f(x) = \sqrt{x}$.

$$\begin{aligned}
f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\
&= \lim_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \cdot \frac{\sqrt{z} + \sqrt{x}}{\sqrt{z} + \sqrt{x}} \\
&= \lim_{z \to x} \frac{\sqrt{z}^2 + \sqrt{z}\sqrt{x} - \sqrt{x}\sqrt{z} - \sqrt{x}^2}{(z - x)\left(\sqrt{z} + \sqrt{x}\right)} \\
&= \lim_{z \to x} \frac{z - x}{(z - x)\left(\sqrt{z} + \sqrt{x}\right)} \\
&= \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \quad = \quad \frac{1}{2\sqrt{x}}
\end{aligned}$$

Therefore $f'(x) = \dfrac{1}{2\sqrt{x}}$

4. (6 points) Find all x for which the tangent to the graph of $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 2x + 1$ has slope 10.

We need to solve the following equation.

$$\begin{aligned}
y' &= 10 \\
x^2 + x - 2 &= 10 \\
x^2 + x - 12 &= 0 \\
(x - 3)(x + 4) &= 0
\end{aligned}$$

Thus the slope equals 10 at $x = 3$ and $x = -4$.

5. (6 points) Suppose it costs $C(x)$ dollars to build a transmitting tower that is x meters high. Suppose it happens that $C'(100) = 1000$. Explain in simple terms what this means.

$C'(x)$ is the rate of change in (dollars per meter) of the cost of building the tower x meters high.

The statement $C'(100) = 1000$ means that when the tower is 100 meters high (i,e., when x=100), the cost is changing at a rate of \$1000 per meter. At this rate it will cost an extra \$1000 to build the tower one additional meter higher.

6. (35 points) Find the derivatives of these functions. You do **not** need to simplify your answers.

(a) $f(x) = x^3 + \pi^3$ .... $\boxed{f'(x) = 3x^2}$

(b) $f(x) = \dfrac{4}{\sqrt[3]{x}} = \dfrac{4}{x^{1/3}} = 4x^{-1/3}$ ...... $f'(x) = 4\left(-\dfrac{1}{3}x^{-1/3-1}\right) = -\dfrac{4}{3}x^{-4/3} = -\dfrac{4}{3x^{4/3}} = \boxed{-\dfrac{4}{3\sqrt[3]{x}^4}}$

(c) $f(x) = \cos\left(\dfrac{x+1}{x-1}\right)$ .... $f'(x) = -\sin\left(\dfrac{x+1}{x-1}\right) D_x\left[\dfrac{x+1}{x-1}\right]$

$$= -\sin\left(\frac{x+1}{x-1}\right)\frac{1\cdot(x-1)-(x+1)\cdot 1}{(x-1)^2}$$

$$= \boxed{2\sin\left(\frac{x+1}{x-1}\right)\frac{1}{(x-1)^2}}$$

(d) $f(x) = \ln|x|\cdot\sec(x)$.................................. $\boxed{f'(x) = \dfrac{1}{x}\sec(x) + \ln(x)\sec(x)\tan(x)}$

(e) $f(x) = \left(\sin^{-1}(x)\right)^3$ .... $f'(x) = 3\left(\sin^{-1}(x)\right)^2 D_x\left[\sin^{-1}(x)\right]$

$$= 3\left(\sin^{-1}(x)\right)^2\frac{1}{\sqrt{1-x^2}}$$

$$= \boxed{\frac{3\left(\sin^{-1}(x)\right)^2}{\sqrt{1-x^2}}}$$

(f) $f(x) = \dfrac{1}{x^2+1} = \left(x^2+1\right)^{-1}$ .... $f'(x) = -\left(x^2+1\right)^{-2}(2x+0) = \boxed{\dfrac{-2x}{\left(x^2+1\right)^2}}$

(g) $y = x\ln\left(\sec\left(x^3+x\right)\right)$ .... $y' = 1\cdot\ln\left(\sec\left(x^3+x\right)\right) + x\,D_x\left[\ln\left(\sec\left(x^3+x\right)\right)\right]$

$$= \ln\left(\sec\left(x^3+x\right)\right) + x\frac{D_x\left[\sec\left(x^3+x\right)\right]}{\sec\left(x^3+x\right)}$$

$$= \ln\left(\sec\left(x^3+x\right)\right) + x\frac{\sec\left(x^3+x\right)\tan\left(x^3+x\right)(3x^2+1)}{\sec\left(x^3+x\right)}$$

$$= \boxed{= \ln\left(\sec\left(x^3+x\right)\right) + x\tan\left(x^3+x\right)(3x^2+1)}$$

7. (7 points) Given the equation $y\ln(x) + y^2 = 5x$, find y'.

$$\begin{aligned} y\ln(x) + y^2 &= 5x \\ D_x\Big[y\ln(x) + y^2\Big] &= D_x\Big[5x\Big] \\ y'\ln(x) + y\frac{1}{x} + 2yy' &= 5 \\ y'\ln(x) + 2yy' &= 5 - \frac{y}{x} \\ y'\big(\ln(x) + 2y\big) &= 5 - \frac{y}{x} \\ y' &= \frac{5 - \frac{y}{x}}{\ln(x) + 2y} \end{aligned}$$

8. (6 points) A spherical balloon is inflated at a rate of 100π cubic feet per minute. How fast is the radius increasing at the instant the radius is 5 feet?

Let V be the balloon's volume and let r be its radius.

Know: $\dfrac{dV}{dt} = 100\pi$ cubic feet per minute.

Want: $\dfrac{dr}{dt}$ at the instant $r = 5$.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ D_t\Big[V\Big] &= D_t\Big[\frac{4}{3}\pi r^3\Big] \\ \frac{dV}{dt} &= \frac{4}{3}3\pi r^2\frac{dr}{dt} \\ \frac{dV}{dt} &= 4\pi r^2\frac{dr}{dt} \\ 100\pi &= 4\pi r^2\frac{dr}{dt} \\ \frac{100\pi}{4\pi r^2} &= \frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{25}{r^2} \end{aligned}$$

Answer: When $r = 5$ the radius is changing at a rate of $\left.\dfrac{dr}{dt}\right|_{r=5} = \dfrac{25}{5^2} = \boxed{\text{1 foot per minute}}$