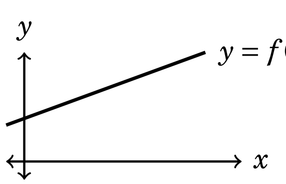
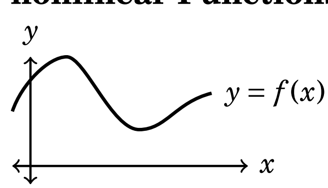
CHAPTER 1

What Calculus is About

All of Calculus is based on one very simple but very far reaching idea. We will get to this big idea in just one page, but first a few words about functions. A function is a mathematical construction that models how one variable (say x) influences another variable (say y). We might write $y = f(x)$ to indicate that for each quantity x there is a corresponding quantity $y = f(x)$. A function can be expressed algebraically. For example, $f(x) = x^2 + 3x + 1$ is a function that gives an output $y = f(x) = x^2 + 3x + 1$ for each input x. In this case the quantity y depends on the quantity x.

Functions are significant because every scientific discipline invloves situations in which one quantity depends upon another. In physics, the gravitational force between two bodies depends on the distance between them. In economics, a commodity's price depends on demand; as demand increases, there is an increase in price (assuming supply is constant).

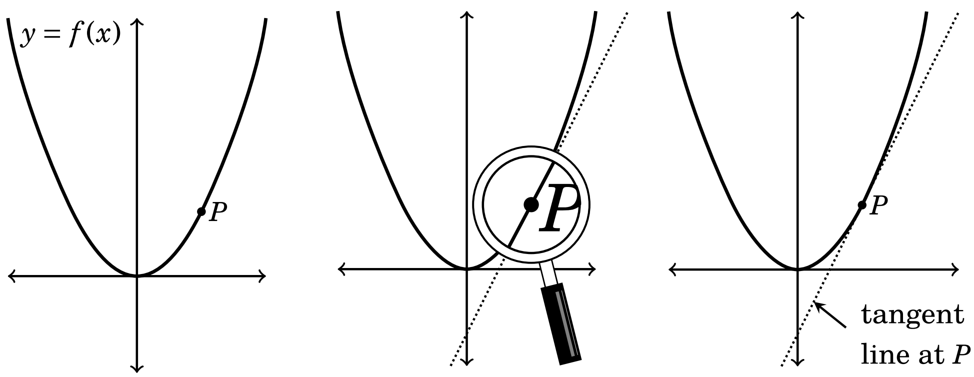
We divide functions into two groups, linear and nonlinear. Linear functions are those that have the simple expression $f(x) = mx + b$; the graph of such a function is a straight line with slope m and y-intercept b. Any other function is what we call nonlinear. Its graph is curved and its algebraic expression might be quite complex.

Linear Functions $f(x) = mx+b$	**nonlinear Functions**
	
Nice properties:	Not-so-nice properties:
• Simple algebraic expression	• Potentially complex expression
• Easy to graph	• Can be hard to graph
• Easy to find intercepts	• Intercepts can be hard to find
• Has a slope	• No immediate notion of slope
• y changes at a constant rate	• y changes at a varying rate

Of course the most useful functions – the ones that arise naturally in scientific disciplines – are not likely to be linear. The physical world (as well as the mathematical universe) is complex, and the functions that describe it are likewise complex.

This presents what seems to be a dilemma: Linear functions are simple, but the most useful functions are nonlinear.

But there is a simple and profound resolution to this dilemma. It is the big idea behind calculus. To understand it, take a nonlinear function like the one graphed on the left below. Pick any point P on its graph. Imagine taking a powerful magnifying glass and looking closely at the part of the graph near P. Up close, the graph looks linear because it doesn't have much opportunity to bend in the small portion we're looking at. Increasing the magnification only makes the graph look straighter.



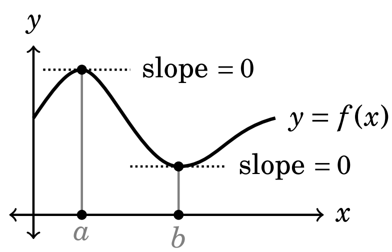
Extending this straight line, we get the picture above on the right. This line though P is called the **tangent line** to the graph at the point P. The tangent line to the curve at a point P is the line through P that has the same direction as the curve at P. (Admittedly this definition is more intuitive than precise, but intuition often gets us further than precision.) This brings us to the very simple, very profound idea that all of calculus is based on.

Main idea of calculus: Up close, nonlinear functions look linear.

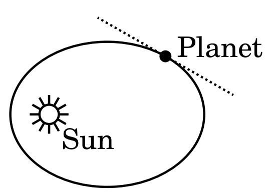
This means that even the most complicated function is potentially very simple: up close it is indistinguishable from its tangent line. Up close its graphs looks linear, and therefore the graph has a slope at each point P, though the slope may differ from point to point. One of the main tools of calculus (as we shall see) is an idea called the *derivative* of a function $f(x)$. The derivative will tell us the slope of the graph of $y = f(x)$ at any point.

The ability to compute slopes of tangent lines has many, many applications. We will close this chapter with just two, though you will encounter many more in this course, and beyond.

One application involves finding optimal outcomes. If $f(x)$ is some desirable quantity (like profit) that depends on some quantity x, we would be interested in finding what value of x makes $f(x)$ as big as possible. On the other hand, if $f(x)$ were some undesirable quantity (like cost) we would be interested in what value of x makes $f(x)$ as small as possible. The high a low points of $f(x)$ happen where its slope is zero, as the below diagram suggests. Being able to find the slopes of tangent lines will allow us to find the x values a and b for which $f(x)$ us at a maximum or minimum.



Another applications involves planetary motion. In fact, Isaac Newton invented calculus in the 1600's as a means of describing motion of planets. If a planet were moving through space without being subject to any external force (such as gravity) it would continue at a uniform motion in a straight line. But in reality it is subject to the gravitational attraction of the sun. This force pulls its trajectory away from the straight line. Calculus can be used to show that gravity forces the planet to move in an elliptical orbit around the sun. When the planet is at a point P on the ellipse, the tangent line through P is the straight line that the planet would move on if the gravity could suddenly "turned off."



Although we will not work out the details of planetary motion in this course, we will see how calculus can be used to solve various problems involving motion.

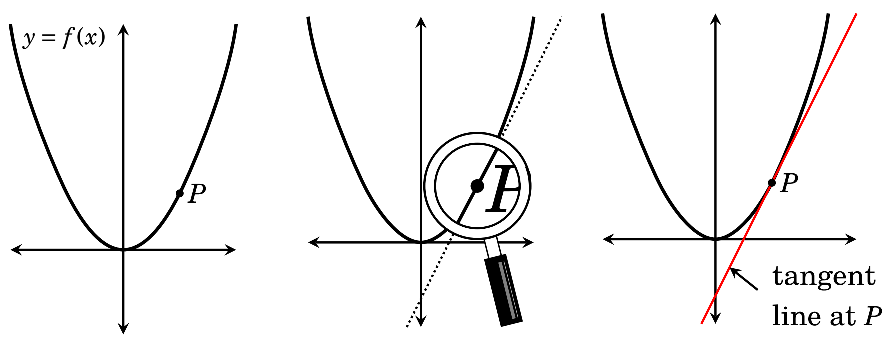
Our discussion so far suggests that calculus is built on the notion of a function. Thus, before discussing calculus in greater detail we will first review functions.

CHAPTER 7

Limits: The Way to Tangent Lines

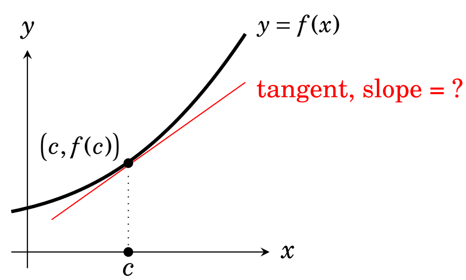
Now that we have reviewed the fundamental idea of a function, we turn to the central problem of calculus: finding slopes of tangent lines.

Recall that, up close, non-linear functions tend to look linear. Take a point P on the graph of a function $f(x)$ and magnify the graph at P. The graph looks like a straight line there because the curve hasn't had much space to bend. The higher the magnification, the straighter the curve looks.



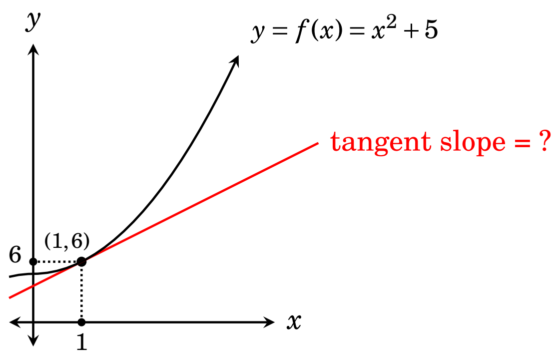
If we were to extend this apparent straight line through P we'd get what is called the *tangent line* to the curve at P. Near P, it touches the graph only at P. Think of it as the best straight line "fit" to the curve at P. At P it has the same direction as the curve.

The **fundamental problem of calculus** is to compute the slope of a tangent line to the graph of a function $y = f(x)$ at a point $(c, f(c))$.



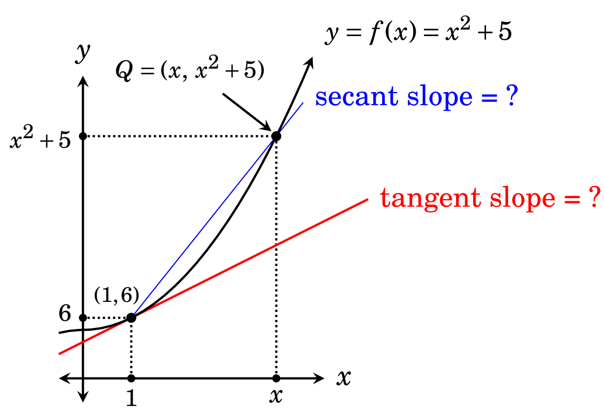
In what follows we will solve this problem for a specific point on a specific function. The solution involves a new mathematical idea called a *limit*. The rest of Part II will be a careful study of this important concept.

Our motivational example concerns the function $f(x) = x^2 + 5$. Its graph is the parabola $y = x^2$ moved up 5 units. On it is the point $(1, f(1)) = (1,6)$. Our goal is to find the slope of the tangent to the graph at $(1,6)$.



To compute its slope we need two points on the line so we can work out rise over run. Unfortunately we know of only one definite point on the tangent line, namely $(1,6)$. We need a second.

We will do the best we can do with the information given. The second point will be not on the tangent line to the curve, but on the curve itself. Take a value of x that is near 1. To x there corresponds a point $Q = (x, f(x)) = (x, x^2 + 5)$ *on the curve*. Draw a line through $(1,6)$ and Q. This new line is not our tangent line because it crosses the graph twice. A line—like this one—crossing a curve at two (or more) points is called a **secant line** to the curve.

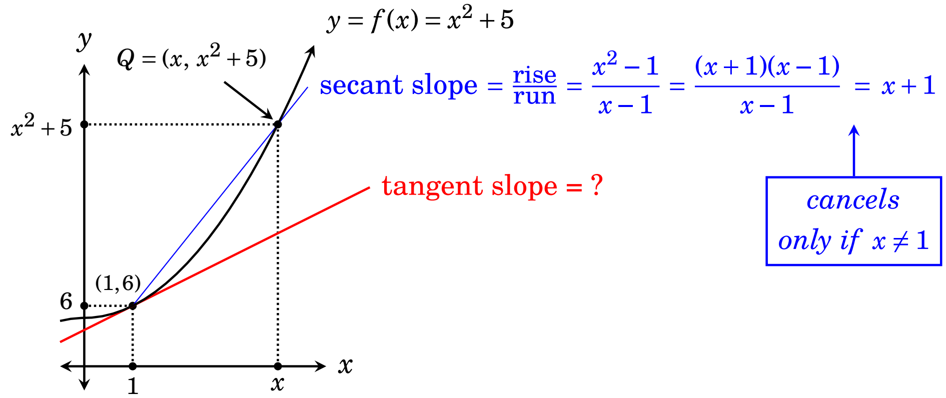


If x is fairly close to 1, the secant line has roughly the same slope as the tangent line. But (unlike the tangent line) we can compute the slope of the secant from the two points $(1,6)$ and $Q = (x, x^2 + 5)$. We get

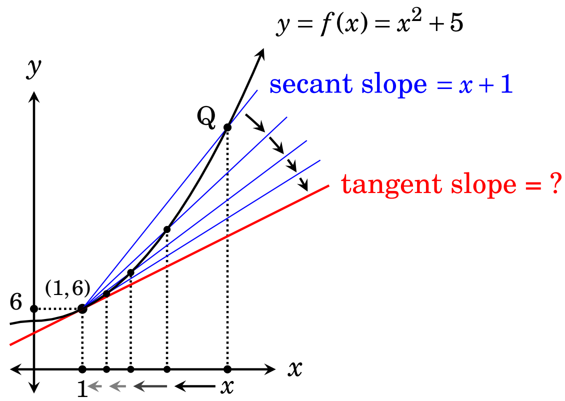
$$\text{secant slope} = \frac{\text{rise}}{\text{run}} = \frac{(x^2+5)-6}{x-1} = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1.$$

Notice that this secant slope depends on x because the location of Q depends on x. Notice also that our set-up relies on $x \neq 1$ because if $x = 1$ the point Q coincides with $(1,6)$ and we no longer have two points to form the secant.

Also, in simplifying the expression for secant slope, the cancelation $\frac{(x+1)(x-1)}{x-1} = x+1$ works only when $x \neq 1$. The reason is that in canceling we are relying on $\frac{x-1}{x-1} = 1$. If $x = 1$ then this fraction is $\frac{0}{0}$, which is not defined.



Of course the secant line, whose slope is $x+1$, is not the tangent line. But if x is very close to 1, the secant line is a reasonable approximation of the tangent line, and the closer x is to 1, the better the approximation.



Imagine x getting closer and closer to the number 1. This makes the point Q move down the curve, getting closer and closer to the point $(1,6)$. The secant line pivots on the point $(1,6)$, rotating toward the tangent line. Thus as x approaches 1, the secant slope $x+1$ approaches the tangent slope. So as x approaches 1, the secant slope $x+1$ approaches $1+1 = 2$, which must be the tangent slope. Now we have our answer.

Conclusion: The tangent to $f(x) = x^2+5$ at $(1,6)$ has slope **2**.

In summary, the tangent slope is the value that the secant slope $\frac{x^2-1}{x-1}$ approaches as x approaches 1. To underscore this point we tally the information into the table below. The first column contains values of x getting closer and closer to 1. The second column shows the corresponding secant slope $\frac{x^2-1}{x-1}$. We can see that as x approaches 1, the corresponding value $\frac{x^2-1}{x-1}$ approaches 2.

x	secant slope $= \dfrac{x^2-1}{x-1} = x+1$
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999
$\downarrow$	$\downarrow$
1	**2**
$\uparrow$	$\uparrow$
1.0001	2.0001
1.001	2.001
1.01	2.01
1.1	2.1
1.5	2.5
2	3

Mathematics has a special notation for the kind of *limiting process* expressed by the table, where x approaching 1 forces $\frac{x^2-1}{x-1}$ to approach 2. We write it as

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2,$$

where the symbol $\lim\limits_{x\to 1}$ means that x is getting closer and closer to 1. We call this expression a **limit**. From this discussion we see that in general the problem of finding the slope of a tangent line to the graph of a function can be solved by a limit of the form

$$\lim_{x\to c} g(x) = L.$$

The picture that emerges from our discussion is this: Finding slopes of tangent lines involves understanding limits. Consequently all of Part 2 of this book is devoted to limits. Once we have a thorough understanding of limits we will return to slopes of tangent lines in Part 3.