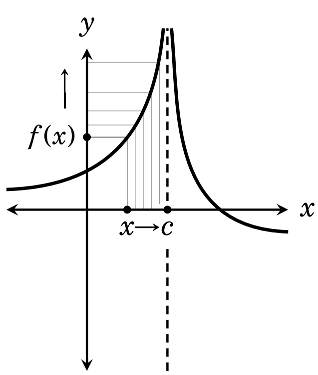
CHAPTER 12

Infinite Limits

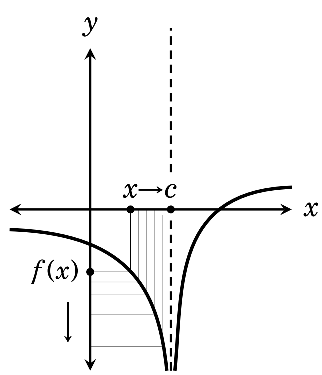
Occasionally a limit does not exist, but it does not exist for a very special reason: as x approaches c, the corresponding value $f(x)$ does not approach any number, but just gets larger and larger without bound. In such a case we say that the limit equals ∞ or $-\infty$. Let's look at some of the ways this can happen.

Below is a function that becomes larger and larger as x gets closer and closer to c. We express this as $\lim\limits_{x\to c} f(x) = \infty$.



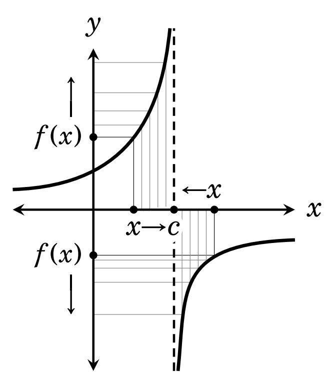
$$\lim_{x\to c} f(x) = \infty$$

Next is an $f(x)$ that becomes infinitely big in the *negative* direction as x gets closer and closer to c. We express this as $\lim\limits_{x\to c} f(x) = -\infty$.



$$\lim_{x\to c} f(x) = -\infty$$

Sometimes right- and left-hand limits are necessary to describe what happens to $f(x)$ at c. Below we have $\lim\limits_{x\to c^+} f(x) = -\infty$ and $\lim\limits_{x\to c^-} f(x) = \infty$.



$$\lim_{x\to c^-} f(x) = \infty$$

$$\lim_{x\to c^+} f(x) = -\infty$$

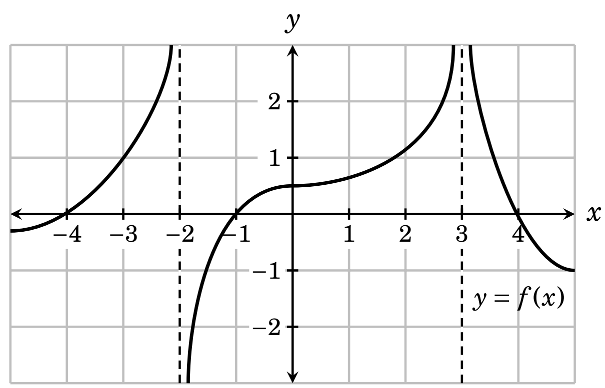
Limits of this type are called **infinite limits**. Technically speaking, such a limit does not exist (DNE) because it does not equal a number. However, saying that the limit is ∞ or $-\infty$ expresses useful information about *how* it does not exist. Thus we will freely treat infinite limits as if they *do* exist, and supply ∞ or $-\infty$ as the answer.

Definition 12.1 If f is a function and c is a number, then

- $\lim_{x\to c} f(x) = \infty$ means $f(x)$ approaches ∞ as x approaches c.
- $\lim_{x\to c} f(x) = -\infty$ means $f(x)$ approaches $-\infty$ as x approaches c.

In either case we say the limit is **infinite**. We also allow for the usual left- and right-hand versions of infinite limits.

For the function graphed below, $\lim_{x\to 3} f(x) = \infty$, but $\lim_{x\to -2} f(x)$ DNE because the answer depends on which direction x approaches 2 from. However, $\lim_{x\to 2^-} f(x) = \infty$ and $\lim_{x\to -2^+} f(x) = -\infty$. (Of course this function has plenty of non-infinite limits, such as $\lim_{x\to 0} f(x) = 0.5$, etc.)



The dashed lines in the above diagram are called *vertical asymptotes*. They are visual aids that help us "hang" the graph of $f(x)$ on the grid in a way that visually expresses the presence of infinite limits. For instance, $f(3)$ is not defined, so there is no point $(3, f(3))$ on the graph above or below $x = 3$. We might be tempted to say the "point" $(3, \infty)$ is on the graph, but there is no such point on the grid. In short, the graph of $f(x)$ simply does not cross the vertical asymptote through $x = 3$. But no matter how close x is to 3, there is a point $(x, f(x))$ on the graph that is very high up, and very close to the vertical asymptote.

If a function has a vertical asymptote, its graph gets arbitrarily close to the asymptote, but becomes higher and higher (or lower and lower) as it does so.

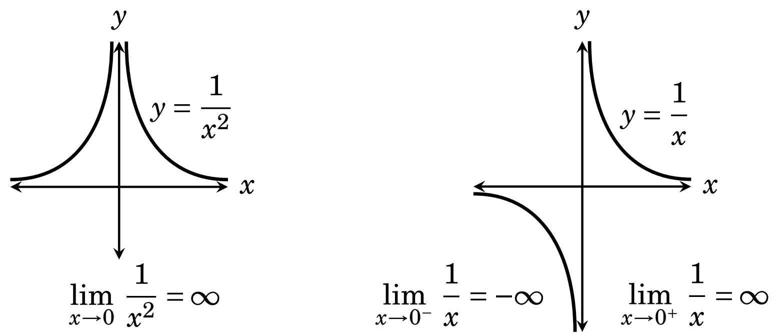
This chapter is concerned with two main questions:

- Given a limit $\lim_{x\to c} f(x)$, how can determine that it equals ∞ or $-\infty$?
- How do we know if a function has any vertical asymptotes? If it does, how can we find them?

We'll also summarize our techniques for computing limits, finite or infinite.

12.1 Recognizing Infinite Limits

Two simple examples will help us understand almost all infinite limits. The first is $\lim_{x\to 0} \frac{1}{x^2}$. Here x approaches 0, and the (positive) value x^2 shrinks to 0, so its reciprocal $\frac{1}{x^2}$ grows bigger and bigger without bound. We conclude $\lim_{x\to 0} \frac{1}{x^2} = \infty$. (See diagram, below left.)



The second example involves $y = \frac{1}{x}$, graphed above. Consider $\lim_{x\to 0} \frac{1}{x}$. If x is very close to zero and *positive*, then $1/x$ is very large and *positive*. (Examples: If $x = 0.01$, then $1/x = 100$, if $x = 0.0001$, then $1/x = 10000$, etc.) But if x is very close to zero and *negative*, then $\frac{1}{x}$ is very large and *negative*. (Examples: If $x = -0.01$, then $1/x = -100$, if $x = -0.0001$, then $1/x = -10000$.) Thus $\lim_{x\to 0} \frac{1}{x}$ depends on which direction x approaches zero from. We have $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

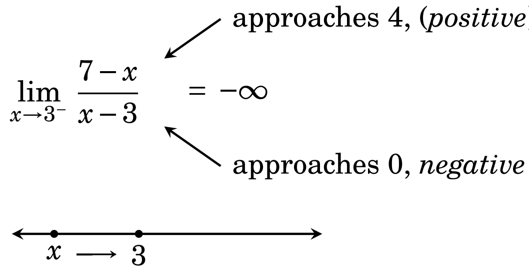
The above examples illustrate how infinite limits typically arise: as limits of fractional expressions in which the numerator is nonzero and the denominator shrinks to zero. this causes the expression to "blow up" to positive or negative infinity. In other words a limit that equals ∞ or $-\infty$ typically has the form $\lim_{x\to c} \frac{f(x)}{g(x)}$, where $f(x)$ becomes close to a non-zero number but $g(x)$ approaches 0. We might get something like this.

$$\lim_{x\to c^+} \frac{\overbrace{f(x)}^{\text{approaching some } \textit{positive} \text{ number}}}{\underbrace{g(x)}_{\text{approaching 0, } \textit{negative}}} = -\infty$$

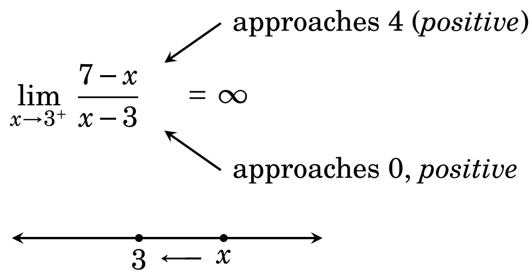
If the top and bottom have the same sign (both positive or both negative), then the answer will be ∞. But if they have opposite signs the answer will be $-\infty$. Let's look at some examples.

Example 12.1 Investigate $\lim\limits_{x\to 3} \dfrac{7-x}{x-3}$.

In this limit the numerator approaches 4 and the denominator shrinks to 0, so we expect an infinite limit. Let's investigate the left- and right-hand limits separately, starting with $\lim\limits_{x\to 3^-} \dfrac{7-x}{x-3}$.

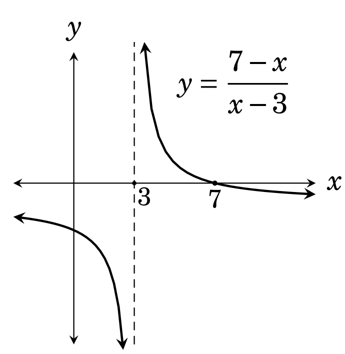


To work this out, first note that the term $7-x$ on top is approaching $7-3=4$. Since x gets close to 3 from the *left*, we can think of it as having a value like $x = 2.9$ or $x = 2.99$, etc., so the $x-3$ on the bottom approaches 0, but is *negative*. (Drawing number line with x to the left of 3 can be helpful; see above.) So in this limit the numerator approaches 4 while the denominator shrinks to 0, but remains negative. As this happens the quotient becomes a larger and larger *negative* number. We conclude $\lim\limits_{x\to 3^-} \dfrac{7-x}{x-3} = -\infty$.



Next, let's look at the right-hand limit. This time x is approaching 3 from the right, so it's larger than 3 (see the number line diagram). Thus $x-3$ is positive, approaching 0. Meanwhile the numerator $7-x$ approaches the positive value 3, so the quotient tends to $+\infty$. $\lim\limits_{x\to 3^+} \dfrac{7-x}{x-3} = \infty$

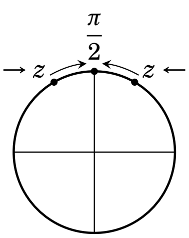
In summary, $\lim\limits_{x\to 3^-} \dfrac{7-x}{x-3} = -\infty$ and $\lim\limits_{x\to 3^+} \dfrac{7-x}{x-3} = \infty$. Because the left- and right-had limits don't agree, $\lim\limits_{x\to 3} \dfrac{7-x}{x-3}$ DNE. Compare this to graph of $y = \dfrac{7-x}{x-3}$. Note vertical asymptote at $x = 3$. ✍



Example 12.2 Determine $\lim_{z\to\frac{\pi}{2}} \dfrac{z}{\cos(z)}$.

In the limit the numerator approaches the positive number $\frac{\pi}{2}$ while the denominator shrinks to $\cos(\pi/2) = 0$. We therefore expect an infinite limit and continue with the left- and right-hand limits. In either case, because z appears as an argument of cos, we know to interpret z as a radian measure. This is illustrated below, with z approaching the radian measure $\frac{\pi}{2}$ (90°) on the unit circle. Notice that, in the right-hand limit $\lim_{z\to\frac{\pi}{2}^+} \dfrac{z}{\cos(z)}$, the number z is *greater* than $\frac{\pi}{2}$, so it is in the second quadrant, and hence $\cos(z)$ is negative. Similarly, in the left-hand limit, $z < \frac{\pi}{2}$, meaning z is in the first quadrant, and $\cos(z)$ is positive.

$\left(\begin{array}{l} z > \frac{\pi}{2}\text{, so } z \text{ is} \\ \text{approaching} \\ \frac{\pi}{2} \text{ from right} \end{array}\right) \to z$



$z \leftarrow \left(\begin{array}{l} z < \frac{\pi}{2}\text{, so } z \text{ is} \\ \text{approaching} \\ \frac{\pi}{2} \text{ from left} \end{array}\right)$

As z approaches $\frac{\pi}{2}$ from the right, the numerator z is positive and the denominator $\cos(z)$ is negative (approaching 0) so the right-hand limit is $-\infty$:

$$\lim_{z\to\frac{\pi}{2}^+} \frac{\overset{\text{approaches } \frac{\pi}{2} \text{ (}\textit{positive}\text{)}}{z}}{\underset{\text{approaches 0, }\textit{negative}}{\cos(z)}} = -\infty$$

As z approaches $\frac{\pi}{2}$ from the left, the numerator z is positive and the denominator $\cos(z)$ is positive (approaching 0) so the right-hand limit is ∞:

$$\lim_{z\to\frac{\pi}{2}^-} \frac{\overset{\text{approaches } \frac{\pi}{2} \text{ (}\textit{positive}\text{)}}{z}}{\underset{\text{approaches 0, }\textit{positive}}{\cos(z)}} = \infty$$

In summary, $\lim_{z\to\frac{\pi}{2}^-} \dfrac{z}{\cos(z)} = \infty$ and $\lim_{z\to\frac{\pi}{2}^+} \dfrac{z}{\cos(z)} = -\infty$, so $\lim_{z\to\frac{\pi}{2}} \dfrac{z}{\cos(z)}$ DNE. ✍

Example 12.3 Determine $\lim_{z\to\pi} \dfrac{1}{1+\cos(x)}$.

Here the numerator is 1 (positive) and the denominator shrinks to $1+\cos(\pi) = 1-1 = 0$. But because $\cos(x) \geq -1$ for any x, the denominator $1+\cos(\pi)$ is positive, no matter how x approaches π. Therefore $\lim_{z\to\pi} \dfrac{1}{1+\cos(x)} = \infty$. ✍

Example 12.4 This example concerns the function $f(x) = \dfrac{x^2-1}{x^3+x^2-x-1}$. Find: (a) $\lim\limits_{x\to 0} f(x)$, (b) $\lim\limits_{x\to 1} f(x)$, and (c) $\lim\limits_{x\to -1} f(x)$.

Solutions: Part (a) is perhaps the easiest, for in this limit the denominator does not approach zero, so we can just apply a familiar limit law.

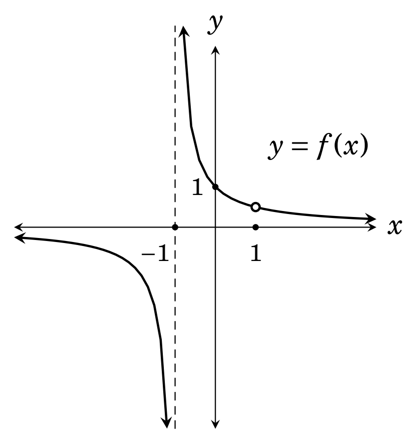
$$\text{(a)}\quad \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2-1}{x^3+x^2-x-1} = \frac{\lim\limits_{x\to 0}\left(x^2-1\right)}{\lim\limits_{x\to 0}\left(x^3+x^2-x-1\right)} = \frac{-1}{-1} = \boxed{1}.$$

In parts (b) and (c), where x is approaching 1 and -1, we have $f(1) = \frac{0}{0}$ and $f(-1) = \frac{0}{0}$, so immediately applying a limit law as we did in part (a) is not going to work. As we well know by now, we need to try to factor and cancel to avoid the $\frac{0}{0}$. So let's factor, cancel and simplify $f(x)$:

$$\begin{aligned} f(x) &= \frac{x^2-1}{x^3+x^2-x-1} = \frac{(x+1)(x-1)}{x^2(x+1)-(x+1)} \\ &= \frac{(x+1)(x-1)}{(x^2-1)(x+1)} = \frac{(x+1)(x-1)}{(x+1)(x-1)(x+1)} = \frac{1}{x+1}. \end{aligned}$$

Now we can quickly dispose of part (b): $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{1}{x+1} = \boxed{\dfrac{1}{2}}$.

Part (c) asks for $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \dfrac{1}{x+1}$. Here the denominator goes to 0 while the numerator remains 1, so we expect an infinite limit. Let's investigate with left- and right-hand limits. In $\lim\limits_{x\to -1^-} f(x)$ the denominator $x+1$ is negative, approaching 0, while the numerator is 1, so $\lim\limits_{x\to -1^-} f(x) = -\infty$. In $\lim\limits_{x\to -1^+} f(x)$ the denominator $x+1$ is positive, approaching 0, so $\lim\limits_{x\to -1^+} f(x) = \infty$.



Since the left- and right-hand limits don't agree, $\lim\limits_{x\to -1} f(x)$ DNE, which answers part (c). The graph of $f(x)$ on the right sheds light on the answers to parts (a), (b) and (c). The graph of $f(x) = \frac{x^2-1}{x^3+x^2-x-1}$ looks just like the graph of $y = \frac{1}{x+1}$ except that it has a hole at $(1, \frac{1}{2})$ because $f(1)$ is not defined. Note how the graph supports our answers $\lim\limits_{x\to 0} f(x) = 1$ and $\lim\limits_{x\to 1} f(x) = \dfrac{1}{2}$. The vertical asymptote at $x = -1$ reinforces the infinite limits at -1. ✍

12.2 Limits of Fractional Expressions: A Summary

As you've no doubt noticed, a majority of the limits we've encountered so far are limits of fractional expressions, like $\lim_{x\to c} \frac{f(x)}{g(x)}$. In working out such a limit you won't always know ahead of time whether it is going to be finite or infinite. Certainly if $\lim_{x\to c} g(x) \neq 0$ we can just apply a limit law to get the finite answer

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

And (as we've seen in this chapter) if $\lim_{x\to c} g(x) = 0$ but $\lim_{x\to c} f(x) \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ will not exist but could be $\pm\infty$. It is nice when things are so clear-cut. But very often you will find yourself in the situation where

$$\frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{0}{0}.$$

In this case you have to apply some cancellation or other cleverness before getting a definitive answer. Here is a summary of how things can play out.

Summary

How to evaluate $\lim_{x\to c} \frac{f(x)}{g(x)}$. (Assuming both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.)

A. If $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$.

B. If $\lim_{x\to c} g(x) = 0$ but $\lim_{x\to c} f(x) \neq 0$, then the limit $\lim_{x\to c} \frac{f(x)}{g(x)}$ **does not exist**. But it (or its left- and right-hand limits) *could* equal ∞ or $-\infty$. Use the techniques outlined in the previous section.

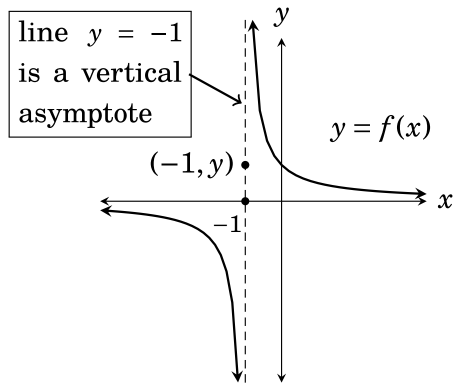
C. If both $\lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} f(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ may or may not exist. To find out, try to discover and cancel the expression that makes $g(x)$ approach zero. The limit should then have one of the forms A or B above. (Most significant limits in calculus are of this type.)

For instance, part (a) of Example 12.4 fell into category A. Parts (b) and (c) both followed category C, but once the simplification was made, one fell into A and the other into B.

12.3 Vertical Asymptotes

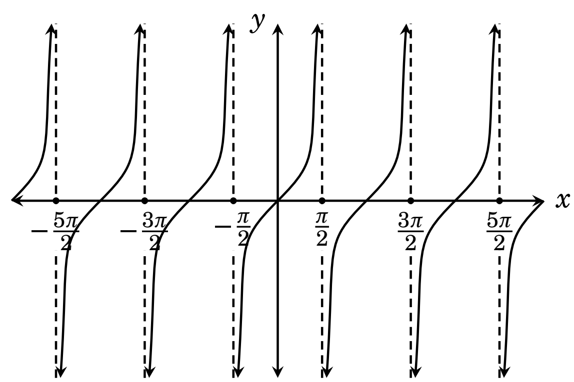
Let's start with a careful definition of a vertical asymptote of a function.

> **Definition 12.2** The line $x = c$ is a **vertical asymptote** of a function f if either $\lim\limits_{x\to c^-} f(x) = \pm\infty$ or $\lim\limits_{x\to c^+} f(x) = \pm\infty$ (or $\lim\limits_{x\to c} f(x) = \pm\infty$).



For example, for the function f from Example 12.4 (graphed again on the right) we found $\lim\limits_{x\to -1^-} f(x) = -\infty$ and $\lim\limits_{x\to -1^+} f(x) = \infty$, so the line $x = -1$ is a vertical asymptote for f. In saying "*the line* $x=-1$" we mean *the line whose equation is* $x = -1$, that is, the set of all points (x, y) on the plane that satisfy the equation $x = -1$. This is the set of all points $(-1, y)$ (where y could be any number), which is a vertical line passing through the x-axis at -1. We might also say that "*f has a vertical asymptote at $x = -1$*'." But please note that a vertical asymptote is a *line* and not a *number*. This line is not actually a part of the graph of f, but it helps us understand and visualize the behavior of f.

A great many functions—such as polynomials and the trig functions $\sin(x)$ and $\cos(x)$—have no vertical asymptotes at all. But when vertical asymptotes are present, they help us understand certain properties of a function. Knowing that the line $x = c$ is a vertical asymptote tells us that even though $f(c)$ may not be defined, the function $f(x)$ "blows up" near c.



Vertical asymptotes tend to happen at numbers $x = c$ that make the denominator of $f(x)$ zero. For example, the function $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$ (graphed on the right) has infinitely many vertical asymptotes, each occurring at a number $x = \frac{k\pi}{2}$ (where k is an integer) that makes the denominator $\cos(x)$ equal to zero.

But the mere fact that a number $x = c$ makes the denominator of $f(x)$ zero does not automatically signal that the line $x = c$ is a vertical asymptote. Definition 12.2 says we must confirm that $\lim\limits_{x\to c^-} f(x) = \pm\infty$ or $\lim\limits_{x\to c^+} f(x) = \pm\infty$.

before saying for sure that the line $x = c$ is a vertical asymptote. Here is a useful rule of thumb for finding vertical asymptotes.

> **How to find the vertical asymptotes (if any) of f(x)**
>
> 1. Identify the values $x = c$ that make the denominator of $f(x)$ equal to zero or at which $f(x)$ if undefined or discontinuous. These are the candidates for the locations of the vertical asymptotes.
> 2. For each c obtained in the previous step, evaluate $\lim\limits_{x\to c^+} f(x)$ or $\lim\limits_{x\to c^-} f(x)$. If you get $\pm\infty$, then the line $x = c$ is a vertical asymptote.

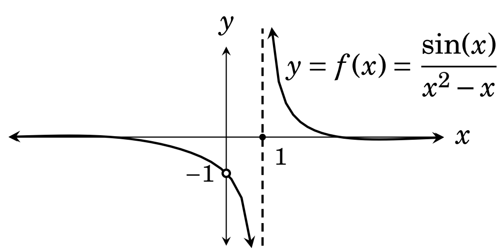
Example 12.5 Find all vertical asymptotes of the function $f(x) = \dfrac{\sin(x)}{x^2 - x}$.

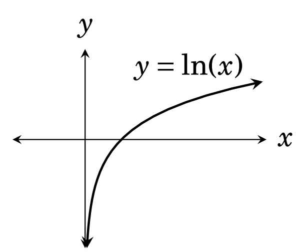
Factoring the denominator yields $f(x) = \dfrac{\sin(x)}{x^2 - x} = \dfrac{\sin(x)}{x(x-1)}$. So the values $x = 0$ and $x = 1$ make the denominator zero, and these are the only two values for which f is undefined. These are our candidates for the locations of vertical asymptotes. Let's first check to see if $x = 0$ yields a vertical asymptote by examining the limit as x approaches 0. Notice that

$$\lim_{x\to 0} \frac{\sin(x)}{x^2 - x} = \lim_{x\to 0} \frac{\sin(x)}{x(x-1)} = \lim_{x\to 0} \frac{\sin(x)}{x} \frac{1}{x-1} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{1}{x-1} = 1 \cdot \frac{1}{0-1} = -1.$$

Since we didn't get $\pm\infty$, there is no vertical asymptote at $x = 0$.

Next let's check the candidate $x = 1$. Consider $\lim\limits_{x\to 1^+} \dfrac{\sin(x)}{x^2 - x}$. The numerator approaches $\sin(1) \approx 0.841 > 0$, while the positive denominator shrinks to 0. Thus $\lim\limits_{x\to 1^+} \dfrac{\sin(x)}{x^2 - x} = \infty$, so the line $x = 1$ is a vertical asymptote. In conclusion, f has only one vertical asymptote, $x = 1$, as shown in the graph below (left). Note the hole at $(0, -1)$, as $f(0)$ is undefined and $\lim\limits_{x\to 0} f(x) = -1$. ✍





Be advised that not every vertical asymptote occurs where the function's denominator is zero. Indeed the function may not even have a denominator yet still have a vertical asymptote. Consider the function $\ln(x)$ (above, right). Our experience with the natural logarithm tells us that $\lim\limits_{x\to 0^+} \ln(x) = -\infty$. so the line $x = 0$ (the y-axis) is a vertical asymptote. Know your functions!

Exercises for Chapter 12

In exercises 1–8 find the limits. (Not every one will be an infinite limit!)

1. $\lim\limits_{x\to 2^+} \dfrac{x^2-x}{x-2}$

2. $\lim\limits_{x\to 3^+} \dfrac{x^2-x}{x^2-9}$

3. $\lim\limits_{x\to 2} \dfrac{x^2-x}{|x-2|}$

4. $\lim\limits_{x\to 1^+} \ln\left(\dfrac{x^2-x}{x-1}\right)$

5. $\lim\limits_{x\to \frac{\pi}{2}^+} \tan(x)$

6. $\lim\limits_{x\to \pi^+} \dfrac{\cos(x)}{1+\cos(x)}$

7. $\lim\limits_{x\to 0^+} \dfrac{x-2}{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)}$

8. $\lim\limits_{x\to 0^+} \dfrac{\sin(x)}{x^2}$

Find the vertical asymptotes of the following functions.

9. $f(x)=\dfrac{x^2-4}{5x^2-10x}$

10. $f(x)=\dfrac{x^2-x-6}{x^2-4x+3}$

11. $f(x)=\dfrac{7x^3-7x^2}{x^2-1}$

12. $f(x)=\dfrac{x^2+3x-15}{x^2+9x+20}$

13. $f(x)=\dfrac{2x^2-8}{x^2+3x+2}$

14. $f(x)=\dfrac{x^2+x-2}{x^2-x-6}$

15. $f(x)=\dfrac{x^2-2x-3}{x^2-1}$

16. $f(x)=\dfrac{x^2+5x+4}{x^2+6x+8}$

17. $f(x)=\dfrac{x^2-1}{7x^3-7x^2}$

18. $f(x)=\dfrac{15-12x-3x^2}{50-2x^2}$

19. $f(x)=\dfrac{x^2+x-6}{2x^2-18}$

20. $f(x)=\dfrac{\cos(x)}{1+\cos(x)}$

21. $f(x)=\tan(x^2)$

22. $f(x)=\dfrac{\sin(x-1)}{x^2-1}$

12.4 Exercises Solutions for Chapter 12

1. $\lim\limits_{x\to 2^+} \dfrac{x^2-x}{x-2} = \infty$ (top approaches 2 and bottom is positive, approaching 0)

3. $\lim\limits_{x\to 2} \dfrac{x^2-x}{|x-2|} = \infty$ (top approaches 2 and bottom is positive, approaching 0)

5. $\lim\limits_{x\to \frac{\pi}{2}^+} \tan(x) = -\infty$ (x in 2nd quadrant of unit circle, so $\tan(x)$ is negative)

7. $\lim\limits_{x\to 0^+} \dfrac{x-2}{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)} = \lim\limits_{x\to 0^+} \dfrac{\sqrt{x}^2-\sqrt{2}^2}{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)} = \lim\limits_{x\to 0^+} \dfrac{\left(\sqrt{x}-\sqrt{2}\right)\left(\sqrt{x}+\sqrt{2}\right)}{\sqrt{x}\left(\sqrt{x}-\sqrt{2}\right)}$

$= \lim\limits_{x\to 0^+} \dfrac{\sqrt{x}+\sqrt{2}}{\sqrt{x}} = \infty$ (top approaching $\sqrt{2}$, bottom positive, approaching 0)

9. $f(x) = \dfrac{x^2-4}{5x^2-10x} = \dfrac{(x-2)(x+2)}{5x(x-2)} = \dfrac{x+2}{5x}$ (for $x \neq 2$).
The denominator is 0 for $x=0$ and $x=2$, so these are the candidates for vertical asymptotes. But $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \frac{x+2}{5x} = \frac{2}{5} \neq \pm\infty$, so there is no vertical asymptote at $x=2$. However, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{x+2}{5x} = \infty$, so $\boxed{\text{the line } x=0 \text{ is a vertical asymptote.}}$

11. $f(x) = \dfrac{7x^3-7x^2}{x^2-1} = \dfrac{7x^2(x-1)}{(x-1)(x+1)} = \dfrac{7x^2}{x+1}$ (for $x \neq 1$).
The denominator is 0 for $x=1$ and $x=-1$, so these are the candidates for vertical asymptotes. But $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \frac{7x^2}{x+1} = \frac{7}{2} \neq \pm\infty$, so no vertical asymptote at $x=1$. However, $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} \frac{7x^2}{x+1} = \infty$, so $\boxed{\text{the line } x=-1 \text{ is a vertical asymptote.}}$

13. $f(x) = \dfrac{2x^2-8}{x^2+3x+2} = \dfrac{2(x-2)(x+2)}{(x+1)(x+2)} = \dfrac{2(x-2)}{x+1}$ (for $x \neq -2$)
The denominator is 0 for $x=-1$ and $x=-2$, so these are the candidates for vertical asymptotes. But $\lim\limits_{x\to -2} f(x) = \lim\limits_{x\to -2} \frac{2(x-2)}{x+1} = 8 \neq \pm\infty$, so no vertical asymptote at $x=-2$. But $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} \frac{2(x-2)}{x+1} = -\infty$, so $\boxed{\text{the line } x=-1 \text{ is a vertical asymptote.}}$

15. $f(x) = \dfrac{x^2-2x-3}{x^2-1} = \dfrac{(x+1)(x-3)}{(x-1)(x+1)} = \dfrac{x-3}{x-1}$ (for $x \neq 2$)
The denominator is 0 for $x=1$ and $x=-1$, so these are the candidates for vertical asymptotes. But $\lim\limits_{x\to -1} f(x) = \lim\limits_{x\to -1} \frac{x-3}{x-1} = 2 \neq \pm\infty$, so no vertical asymptote at $x=-1$. However, $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \frac{x-3}{x-1} = -\infty$, so $\boxed{\text{the line } x=1 \text{ is a vertical asymptote.}}$

17. $f(x) = \dfrac{x^2-1}{7x^3-7x^2}. = \dfrac{(x-1)(x+1)}{7x^2(x-1)} = \dfrac{x+1}{7x^2}$ (for $x \neq 1$).
The denominator is 0 for $x=0$ and $x=1$, so these are the candidates for vertical asymptotes. But $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \frac{x+1}{7x^2} = \frac{2}{7} \neq \pm\infty$, so no vertical asymptote at $x=1$. However, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{x+1}{7x^2} = \infty$, so $\boxed{\text{the line } x=0 \text{ is a vertical asymptote.}}$

19. $f(x) = \dfrac{x^2+x-6}{2x^2-18} = \dfrac{(x-2)(x+3)}{2(x-3)(x+3)} = \dfrac{x-2}{2(x-3)}$ (for $x \neq -3$)

The denominator is 0 for $x=3$ and $x=-3$, so these are the candidates for vertical asymptotes. But $\lim\limits_{x\to-3} f(x) = \lim\limits_{x\to-3} \frac{x-2}{2(x-3)} = \frac{5}{12} \neq \pm\infty$, so no vertical asymptote at $x=-3$. However, $\lim\limits_{x\to3^+} f(x) = \lim\limits_{x\to3^+} \frac{x-2}{2(x-3)} = \infty$, so $\boxed{\text{the line } x=3 \text{ is a vertical asymptote.}}$

21. $f(x) = \tan(x^2)$

The vertical asymptotes happen where $x^2 = \frac{\pi}{2} + k\pi$ for $k = 0, \pm, \pm2, \pm3, \ldots$. Therefore

$\boxed{\text{the vertical asymptotes are the lines } x = \pm\sqrt{\frac{\pi}{2}+k\pi} \text{ for } k = 0, \pm, \pm2, \pm3, \ldots.}$