CHAPTER 15

Slopes of Tangent Lines

We enter Part 3 of the course at a pivotal point. In Chapter 1 we remarked that the fundamental problem of calculus is to find the slope of the tangent line to the graph of a function $f(x)$ at the point $\big(a, f(a)\big)$, as illustrated in Figure 15.1.

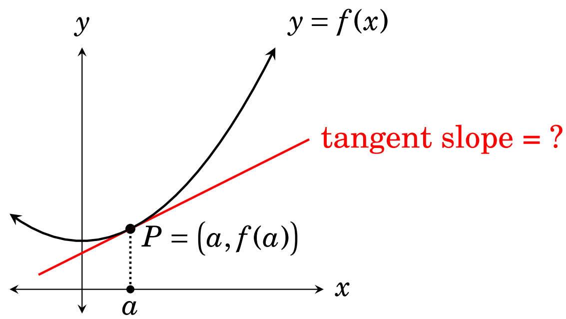


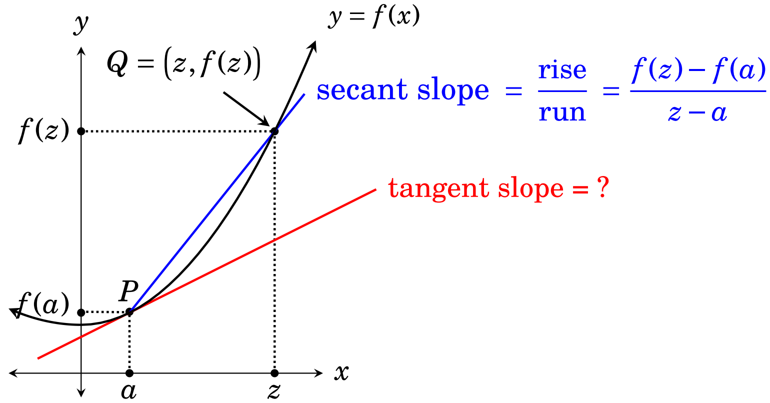
Figure 15.1. The primary problem of calculus is to find the slope of the tangent line to a function $f(x)$ at a point $\big(a, f(a)\big)$.

This goal instigated a review of functions, which we undertook in Part 1. Then, at the beginning or Part 2, we realized that finding slopes of tangent lines involved a new idea, called a *limit*, and we spent all of Part 2 exploring limits. Now, in Part 3, we return to our original problem. Our task is to apply the ideas from Parts 1 and 2 to the problem of finding slopes of tangent lines. More importantly, Part 3 will distill the idea of tangent slopes into the far-reaching idea of what is called the *derivative* of a function. This fundamental idea forms to core of the book, as indicated in the overview below. Later, Part 4 explores the many applications of derivatives, and Part 5 introduces the process of *integration*, which can (as we will see) be viewed as the reverse process of finding derivatives.

Overview of the book

Part 1 Functions → **Part 2** Limits → **Part 3** Derivatives → **Part 4** Applications → **Part 5** Integration

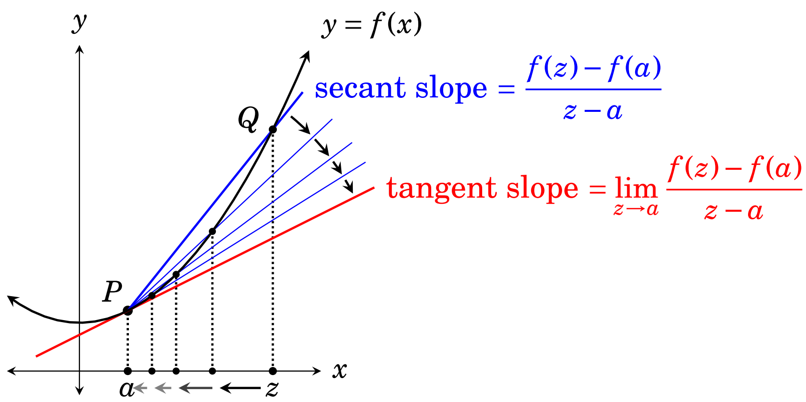
Now we will answer our main question. Given a function $y = f(x)$ and a point $P = \big(a, f(a)\big)$ on its graph, what is the slope of the tangent line at P?



We will attack this problem in the same way as in Chapter 7. Take a number z that is close to a and consider a second point $Q = \big(z, f(z)\big)$, which is on the graph of f. (See the diagram above.) The line through P and Q is called a **secant line**. We can compute its slope from the points P and Q:

$$\text{secant slope} = \frac{\text{rise}}{\text{run}} = \frac{f(z) - f(a)}{z - a}.$$

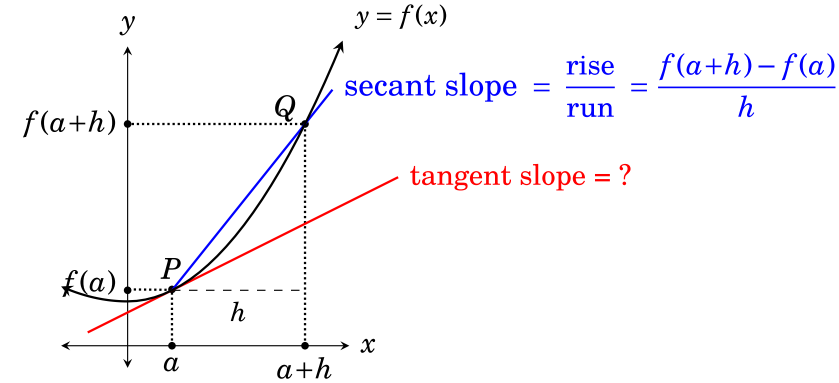
Now imagine z moving in closer and closer to a. As this happens, the point Q moves down the curve, closer and closer to P, and the secant line rotates closer and closer to the tangent line.



Thus as z approaches a, the secant slope approaches the tangent slope, which is to say the tangent slope equals $\lim\limits_{z \to a} \dfrac{f(z) - f(a)}{z - a}$. This is our answer:

Fact 15.1 The tangent to $y = f(x)$ at $\big(a, f(a)\big)$ has slope $\lim\limits_{z \to a} \dfrac{f(z) - f(a)}{z - a}$.

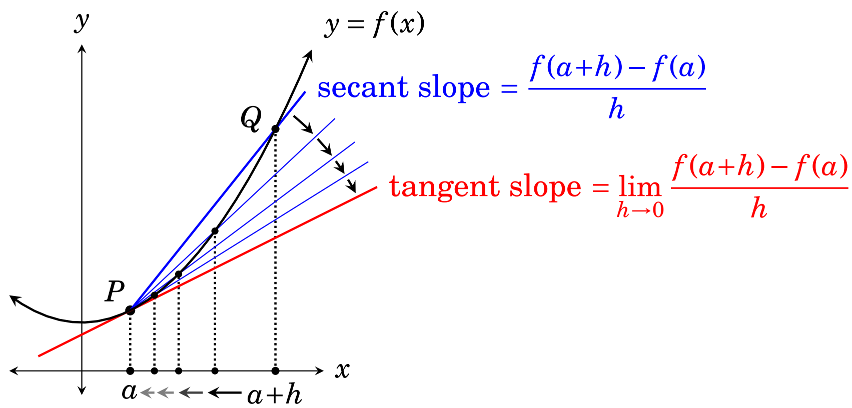
There is a slight variation on this formula for tangent slope that will be useful. Everything is the same except that the number approaching a is called $a+h$ instead of z. (See the diagram below.) The idea is that h is a small number added to a, so $a+h$ is close to a. As before, take a point $Q = (a+h, f(a+h))$ on the curve and consider the secant line through P and Q.



Under this setup, the run between P and Q is $(a+h) - a = h$. Therefore

$$\text{secant slope} = \frac{\text{rise}}{\text{run}} = \frac{f(a+h)-f(a)}{h}.$$

Now imagine h getting smaller and smaller, closer to 0. As this happens, $a+h$ moves closer and closer to a, the point Q moves along the curve towards P, and the secant line pivots at P, rotating closer and closer to the tangent.



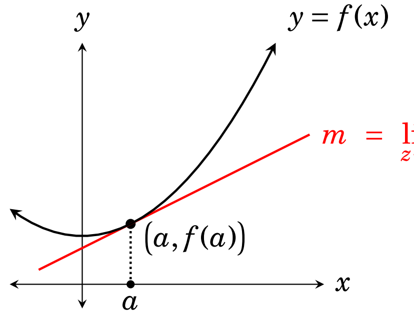
Thus as h approaches 0, the secant slope approaches the tangent slope, which is to say the tangent slope equals $\displaystyle\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Fact 15.2 The tangent to $y = f(x)$ at $\big(a, f(a)\big)$ has slope $\displaystyle\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Facts 15.1 and 15.2 solve the primary problem of finding slopes of tangent lines. Let's recognize this by summarizing them as a theorem.

Theorem 15.1 The slope of the tangent line to the graph of $y = f(x)$ at the point $\big(a, f(a)\big)$ is equal to either of the two limits

$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a} \quad \text{or} \quad \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$



$$m = \lim_{z\to a} \frac{f(z)-f(a)}{z-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

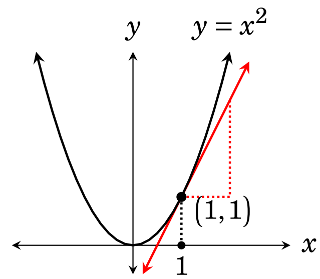
(There is a caveat here. In saying that **the** tangent line has the stated slope, we are assuming that the graph of f actually *has* at tangent line at $\big(a, f(a)\big)$. In some instances there is no tangent line, and in such cases the limits do not exist. More on this later.)

Example 15.1 Find the slope of the tangent to $f(x) = x^2$ at the point $\big(1, f(1)\big) = (1,1)$.

Theorem 15.1 gives two formulas for the slope at $\big(a, f(a)\big)$. In this problem we are interested in the slope at $\big(a, f(a)\big) = \big(1, f(1)\big)$, so we will have $a = 1$ in the formulas. Using the first formula, the slope of the tangent is

$$\begin{aligned}\lim_{z\to a} \frac{f(z)-f(a)}{z-a} &= \lim_{z\to 1} \frac{f(z)-f(1)}{z-1} \\ &= \lim_{z\to 1} \frac{z^2-1^2}{z-1} = \lim_{z\to 1} \frac{(z-1)(z+1)}{z-1} = \lim_{z\to 1}(z+1) = 1+1 = \boxed{2}.\end{aligned}$$

Thus the slope of the tangent to $y = x^2$ at $(1,1)$ is 2. The graph and tangent are shown on the right. Notice that the rise of the tangent does appear to be twice the run, supporting the answer of 2.



We have our answer. But to further illustrate Theorem 15.1, let's use the *second* formula to calculate slope. The slope of the tangent at $(1,1)$ is

$$\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\
&= \lim_{h\to 0}\frac{(1+h)^2-1^2}{h} = \lim_{h\to 0}\frac{1^2+2h+h^2-1}{h} = \lim_{h\to 0}\frac{2h+h^2}{h} \\
&= \lim_{h\to 0}\frac{h(2+h)}{h} = \lim_{h\to 0}(2+h) = 2+0 = \boxed{2}
\end{aligned}$$

We's now calculated the tangent's slope in two ways, using the two formulas of Theorem 15.1. In each case we got a slope of 2, but the first formula entailed a simpler computation. This is typical. Depending on the problem, one of the two formulas may be easier to apply. ✍

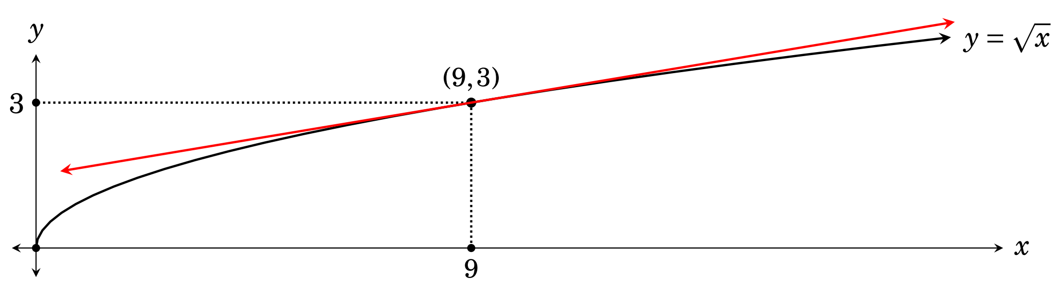
Example 15.2 Find the slope of the tangent to the graph of $y=\sqrt{x}$ at the point $(9,\sqrt{9})=(9,3)$.

The graph and the tangent at $(9,3)$ are shown below. Using the first formula in Theorem 15.1 with $f(x)=\sqrt{x}$ and $a=9$, the tangent slope is

$$\lim_{z\to a}\frac{f(z)-f(a)}{z-a} = \lim_{z\to 9}\frac{\sqrt{z}-\sqrt{9}}{z-9} = \lim_{z\to 9}\frac{\sqrt{z}-3}{z-9}.$$

This is exactly the same limit we worked in Example 9.4 on page 140. As noted there, we are getting a zero in the denominator, but we can cancel the $z-9$ through multiplying by the conjugate of $\sqrt{z}-3$ over itself:

$$\begin{aligned}
\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}\,\frac{\sqrt{x}+3}{\sqrt{x}+3} \\
&= \lim_{x\to 9}\frac{x-9}{(x-9)\left(\sqrt{x}+3\right)} = \lim_{x\to 9}\frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.
\end{aligned}$$



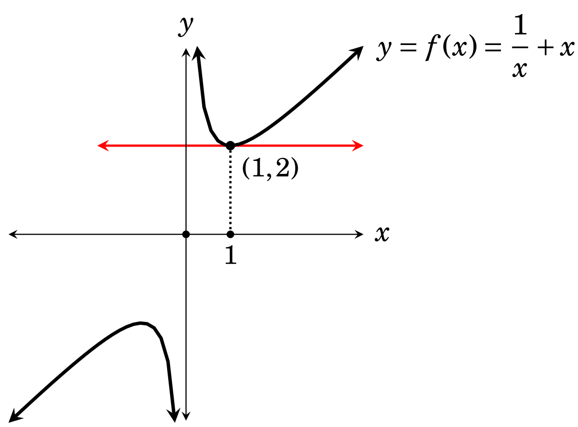
Thus the tangent at $(9,3)$ has slope $\frac{1}{6}$. The alternate formula $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ will give the same answer. ✍

Example 15.3 Find the slope of the tangent to $f(x) = \frac{1}{x} + x$ at $\big(1, f(1)\big)$.

We use the slope formula from Theorem 15.1 with $a = 1$, and use algebra to cancel the resulting $z - 1$ in the denominator.

$$\begin{aligned}
\lim_{z\to a} \frac{f(z)-f(a)}{z-a} &= \lim_{z\to 1} \frac{f(z)\,-\,f(1)}{z-1} \\
&= \lim_{z\to 1} \frac{\left(\frac{1}{z}+z\right)-\left(\frac{1}{1}+1\right)}{z-1} \\
&= \lim_{z\to 1} \frac{\left(\frac{1}{z}+z\right)-2}{z-1} \\
&= \lim_{z\to 1} \frac{\left(\frac{1}{z}+z\right)-2}{z-1}\cdot\frac{z}{z} && \text{(to clear fraction on top)} \\
&= \lim_{z\to 1} \frac{1+z^2-2z}{(z-1)z} && \text{(distribute)} \\
&= \lim_{z\to 1} \frac{z^2-2z+1}{(z-1)z} && \text{(rearrange)} \\
&= \lim_{z\to 1} \frac{(z-1)(z-1)}{(z-1)z} && \text{(factor)} \\
&= \lim_{z\to 1} \frac{z-1}{z} && \text{(cancel)} \\
&= \frac{1-1}{1} = \boxed{0} && \text{(limit law)}
\end{aligned}$$

Therefore to slope of the tangent to $f(x)$ at the point $\big(1, f(1)\big) = (1,2)$ is 0.



The graph of $f(x)$ is shown above. The tangent line at $(1,2)$ is horizontal, with slope 0. ✍

Exercises for Chapter 15

Use the techniques of this chapter to answer the following questions.

1. Find the slope of the tangent to $f(x)=x^2$ at the point $(3,9)$.

2. Find the slope of the tangent to $f(x)=x^2$ at the point $(-2,4)$.

3. Find the slope of the tangent to $f(x)=\dfrac{1}{x^2}$ at the point $(1,1)$.

4. Find the slope of the tangent to $f(x)=\dfrac{1}{x^2}$ at the point $\left(-2,\frac{1}{4}\right)$.

5. Find the slope of the tangent to $f(x)=\dfrac{1}{x}$ at the point $\left(2,\frac{1}{2}\right)$.

6. Find the slope of the tangent to $f(x)=\dfrac{1}{x}$ at the point $\left(-2,-\frac{1}{2}\right)$.

7. Find the slope of the tangent to $f(x)=\sqrt{2x}$ at the point $(2,2)$.

8. Find the slope of the tangent to $f(x)=x^3$ at the point $(2,8)$.

9. Find the slope of the tangent to $f(x)=x^2+x$ at the point $(2,6)$.

10. Find the slope of the tangent to $f(x)=x^2+x$ at the point $\left(-\frac{1}{2},-\frac{1}{4}\right)$.

11. Find the slope of the tangent to $f(x)=x^2+x$ at the point $(0,0)$.

12. Find the slope of the tangent to $f(x)=\sqrt{x}-3$ at the point $(16,1)$.

Exercises Solutions for Chapter 15

1. Find the slope of the tangent to $f(x) = x^2$ at the point $(3,9)$.

$$m = \lim_{z\to 3} \frac{f(z)-f(3)}{z-3} = \lim_{z\to 3} \frac{z^2-3^2}{z-3} = \lim_{z\to 3} \frac{(z-3)(z+3)}{z-3} = \lim_{z\to 3}(z+3) = 3+3 = 6$$

3. Find the slope of the tangent to $f(x) = \dfrac{1}{x^2}$ at the point $(1,1)$.

$$m = \lim_{z\to 1} \frac{f(z)-f(1)}{z-1} = \lim_{z\to 1} \frac{\frac{1}{z^2}-\frac{1}{1^2}}{z-1} = \lim_{z\to 1} \frac{\frac{1}{z^2}-\frac{1}{1^2}}{z-1}\frac{z^2}{z^2} = \lim_{z\to 1} \frac{1-z^2}{(z-1)z^2} = \lim_{z\to 1} \frac{(1-z)(1+z)}{(z-1)z^2}$$
$$= \lim_{z\to 1} \frac{-(1+z)}{z^2} = \frac{-(1+1)}{1^2} = -2$$

5. Find the slope of the tangent to $f(x) = \dfrac{1}{x}$ at the point $\left(2, \frac{1}{2}\right)$.

$$m = \lim_{z\to 2} \frac{f(z)-f(2)}{z-2} = \lim_{z\to 2} \frac{\frac{1}{z}-\frac{1}{2}}{z-2} = \lim_{z\to 2} \frac{\frac{1}{z}-\frac{1}{2}}{z-2}\cdot\frac{2z}{2z} = \lim_{z\to 2} \frac{2-z}{(z-2)2z} = \lim_{z\to 2} \frac{-1}{2z} = -\frac{1}{4}$$

7. Find the slope of the tangent to $f(x) = \sqrt{2x}$ at the point $(2,2)$.

$$m = \lim_{z\to 2} \frac{f(z)-f(2)}{z-2} = \lim_{z\to 2} \frac{\sqrt{2z}-\sqrt{2\cdot 2}}{x-2} = \lim_{z\to 2} \frac{\sqrt{2z}-2}{z-2} = \lim_{z\to 2} \frac{\sqrt{2z}-2}{z-2}\cdot\frac{\sqrt{2z}+2}{\sqrt{2z}+2}$$
$$= \lim_{z\to 2} \frac{2z-4}{(z-2)\left(\sqrt{2z}+2\right)} = \lim_{z\to 2} \frac{2(z-2)}{(z-2)\left(\sqrt{2z}+2\right)} = \lim_{z\to 2} \frac{2}{\sqrt{2z}+2} = \frac{2}{\sqrt{2\cdot 2}+2} = \frac{1}{2}$$

9. Find the slope of the tangent to $f(x) = x^2 + x$ at the point $(2,6)$.

$$m = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\left((2+h)^2+(2+h)\right)-(2^2+2)}{h} = \lim_{h\to 0} \frac{4+4h+h^2+2+h-6}{h}$$
$$= \lim_{h\to 0} \frac{h(5+h)}{h} = \lim_{h\to 0}(5+h) = 5$$

11. Find the slope of the tangent to $f(x) = x^2 + x$ at the point $(0,0)$.

$$m = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\left((0+h)^2+(0+h)\right)-(0^2+0)}{h} = \lim_{h\to 0} \frac{h^2+h}{h} = \lim_{h\to 0} \frac{h(h+1)}{h}$$
$$= \lim_{h\to 0}(h+1) = 1$$