CHAPTER 20

The Product and Quotient Rules

We have developed rules for the derivatives of the sum or difference of two functions that work as follows

$$D_x\Big[f(x)+g(x)\Big] = f'(x)+g'(x)$$

$$D_x\Big[f(x)-g(x)\Big] = f'(x)-g'(x).$$

But what about the derivative of product of two functions, or the quotient of two functions? This chapter answers these questions by deriving two new rules for

$$D_x\Big[f(x)\cdot g(x)\Big] \quad \text{and}$$

$$D_x\left[\frac{f(x)}{g(x)}\right].$$

These two new rules will be called the *product rule* and the *quotient rule*, respectively.

Let's begin by deriving the product rule. Given two functions $f(x)$ and $g(x)$, we aim to work out the derivative of their product, that is, $D_x\Big[f(x)g(x)\Big]$. By Definition 16.1, the derivative of a function $F(x)$ is

$$D_x\Big[F(x)\Big] = \lim_{z\to x}\frac{F(z)-F(x)}{z-x}.$$

We are interested in the case where $F(x) = f(x)g(x)$, which is

$$D_x\Big[f(x)g(x)\Big] = \lim_{z\to x}\frac{f(z)g(z)-f(x)g(x)}{z-x}.$$

We will now work this limit out carefully. In this limit, the denominator $z-x$ approaches zero, so we have to get rid of it somehow. In the following computation it gets absorbed into the definition of $f'(x)$ and $g'(x)$.

So let us begin our computation. As noted above, our first step is

$$D_x\Big[f(x)g(x)\Big] = \lim_{z\to x}\frac{f(z)g(z)-f(x)g(x)}{z-x}.$$

Let's insert a little space int this expression to give ourselves room to work.

$$D_x\Big[f(x)g(x)\Big] = \lim_{z\to x}\frac{f(z)g(z)\qquad\qquad\qquad\qquad -f(x)g(x)}{z-x}.$$

To the space just created, add zero in the form of $0=-f(x)g(z)+f(x)g(z)$. This is an allowable step because adding in 0 doesn't alter the limit's value.

$$D_x\Big[f(x)g(x)\Big] = \lim_{z\to x}\frac{f(z)g(z)-f(x)g(z)\;+\;f(x)g(z)-f(x)g(x)}{z-x}.$$

In the numerator, factor out $g(z)$ from the first two terms, and $f(x)$ from the second two, as shown below. Then split the fraction and apply limit laws:

$$\begin{aligned}
D_x\Big[f(x)g(x)\Big] &= \lim_{z\to x}\frac{\big(f(z)-f(x)\big)g(z)\;+\;f(x)\big(g(z)-g(x)\big)}{z-x}\\
&= \lim_{z\to x}\left(\frac{f(z)-f(x)}{z-x}g(z)\;+\;f(x)\frac{g(z)-g(x)}{z-x}\right)\\
&= \lim_{z\to x}\frac{f(z)-f(x)}{z-x}\cdot\lim_{z\to x}g(z)\;+\;\lim_{z\to x}f(x)\cdot\lim_{z\to x}\frac{g(z)-g(x)}{z-x}\\
&= \boxed{f'(x)g(x)+f(x)g'(x)}.
\end{aligned}$$

In the last step we used the facts that $\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=f'(x)$, $\lim_{z\to x}g(z)=g(x)$, $\lim_{z\to x}f(x)=f(x)$, and $\lim_{z\to x}\frac{g(z)-g(x)}{z-x}=g'(x)$. (We assume here that $g'(x)$ exists, so, by Theorem 18.1, $g(x)$ is continuous, and hence $\lim_{z\to x}g(z)=g(x)$.)

We have just determined that $D_x\big[f(x)g(x)\big]=f'(x)g(x)+f(x)g'(x)$. Now that we have done this computation—and we believe it—we will never have to do it again. It becomes our latest rule.

Rule 7 (The Product Rule) $D_x\Big[f(x)g(x)\Big]=f'(x)g(x)+f(x)g'(x)$

The derivative of a product equals the derivative of the first function times the second function, plus the first function times the derivative of the second:

$$D_x\Big[f(x)g(x)\Big]=D_x\Big[f(x)\Big]\cdot g(x)\;+\;f(x)\cdot D_x\Big[g(x)\Big].$$

In applying the product rule to $f(x)g(x)$, you also have to do the derivatives of f and g, using whatever rules apply to them.

Example 20.1 Find the derivative of $4x^3e^x$.

This is a product $(4x^3)\cdot(e^x)$ of two functions, so we use the product rule.

$$\begin{aligned} D_x\left[4x^3e^x\right] &= D_x\left[4x^3\right]\cdot e^x + 4x^3\cdot D_x\left[e^x\right] \\ &= 12x^2\cdot e^x + 4x^3\cdot e^x \\ &= \boxed{4e^x\left(3x^2+x^3\right)}. \end{aligned}$$

Example 20.2 Find the derivative of $y = \left(x^2+3\right)\left(x^2+5x-7\right)$.

This is a product of two functions, so we use the product rule.

$$\begin{aligned} D_x\left[\left(x^2+3\right)\left(x^2+5x-7\right)\right] &= D_x\left[x^2+3\right]\cdot\left(x^2+5x-7\right) + \left(x^2+3\right)\cdot D_x\left[x^2+5x-7\right] \\ &= 2x\left(x^2+5x-7\right) + \left(x^2+3\right)(2x+5) \end{aligned}$$

This is the derivative, but some simplification is possible:

$$\begin{aligned} &= 2x^3+10x^2-14x \;+\; 2x^3+5x^2+6x+15 \\ &= \boxed{4x^3+15x^2-8x+15}. \end{aligned}$$

Here our approach was to take the derivative (using the product rule) and then simplify. Alternatively, you could first simplify (by multiplying) and *then* take the derivative:

$$\begin{aligned} D_x\left[\left(x^2+3\right)\left(x^2+5x-7\right)\right] &= D_x\left[x^4+5x^3-7x^2 \;+\; 3x^2+15x-21\right] \\ &= 4x^3+15x^2-14x+6x+15 \\ &= \boxed{4x^3+15x^2-8x+15}. \end{aligned}$$

Example 20.3 Very often you'll have a choice of rules, and one choice may lead to a simpler computation than the other. Consider finding the derivative of $y = 3x^5$. This is a product, so you could use the product rule:

$$D_x\left[3x^5\right] = D_x[3]\cdot x^5 + 3\cdot D_x\left[x^5\right] = 0\cdot x^5 + 3\cdot 5x^4 = \boxed{15x^4}.$$

But it's much easier to just use the constant multiple rule:

$$D_x\left[3x^5\right] = 3D_x\left[x^5\right] = 3\cdot 5x^4 = \boxed{15x^4}.$$

Next we derive the rule for $D_x\left[\frac{f(x)}{g(x)}\right]$. (You may opt to skip the derivation on a first reading and go straight to the conclusion at the bottom of this page.) Our computation begins with the definition of the derivative and proceed by adding zero in a clever way, as we did for the product rule.

$$
\begin{aligned}
D_x\left[\frac{f(x)}{g(x)}\right] &= \lim_{z\to x}\frac{\dfrac{f(z)}{g(z)}-\dfrac{f(x)}{g(x)}}{z-x} && \text{(definition of derivative)}\\
&= \lim_{z\to x}\frac{\left(\dfrac{f(z)}{g(z)}\qquad\right)-\left(\dfrac{f(x)}{g(x)}\qquad\right)}{z-x} && \text{(insert space)}\\
&= \lim_{z\to x}\frac{\left(\dfrac{f(z)}{g(z)}-\dfrac{f(x)}{g(z)}\right)-\left(\dfrac{f(x)}{g(x)}-\dfrac{f(x)}{g(z)}\right)}{z-x} && \text{(add zero)}\\
&= \lim_{z\to x}\frac{\big(f(z)-f(x)\big)\dfrac{1}{g(z)}-f(x)\left(\dfrac{1}{g(x)}-\dfrac{1}{g(z)}\right)}{z-x} && \text{(factor)}\\
&= \lim_{z\to x}\frac{\big(f(z)-f(x)\big)\dfrac{1}{g(z)}-f(x)\left(\dfrac{g(z)-g(x)}{g(x)g(z)}\right)}{z-x} && \text{(combine fractions)}\\
&= \lim_{z\to x}\frac{\big(f(z)-f(x)\big)\dfrac{1}{g(z)}-\dfrac{f(x)}{g(x)g(z)}\big(g(z)-g(x)\big)}{z-x} && \text{(regroup)}\\
&= \lim_{z\to x}\left(\frac{f(z)-f(x)}{z-x}\frac{1}{g(z)}-\frac{f(x)}{g(x)g(z)}\frac{g(z)-g(x)}{z-x}\right) && \text{(split fraction)}\\
&= \lim_{z\to x}\frac{f(z)-f(x)}{z-x}\cdot\lim_{z\to x}\frac{1}{g(z)}-\lim_{z\to x}\frac{f(x)}{g(x)g(z)}\cdot\lim_{z\to x}\frac{g(z)-g(x)}{z-x} && \text{(limit laws)}\\
&= f'(x)\frac{1}{g(x)}-\frac{f(x)}{g(x)g(x)}g'(x) && \text{(evaluate limits)}\\
&= \frac{f'(x)g(x)}{g(x)g(x)}-\frac{f(x)g'(x)}{g(x)g(x)} && \text{(get common denominator)}\\
&= \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}. && \text{(combine fractions)}
\end{aligned}
$$

We have our new rule.

Rule 8 (The Quotient Rule) $D_x\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

Compare the differences and similarities between the product rule and the quotient rule. There are similarities, but quotient rule is more complex. It has a minus instead of a plus, *and* you must divide by $g(x)^2$.

$$\text{Product Rule:} \quad D_x\Big[f(x)g(x)\Big] = D_x\Big[f(x)\Big]\cdot g(x) + f(x)\cdot D_x\Big[g(x)\Big]$$

$$\text{Quotient Rule:} \quad D_x\left[\frac{f(x)}{g(x)}\right] = \frac{D_x\Big[f(x)\Big]\cdot g(x) - f(x)\cdot D_x\Big[g(x)\Big]}{\big(g(x)\big)^2}$$

Example 20.4 Find the derivative of $\dfrac{x^2+1}{x^2-x}$.

This function is a quotient, so we apply the quotient rule to find its derivative.

$$\begin{aligned} D_x\left[\frac{x^2+1}{x^2-x}\right] &= \frac{D_x\left[x^2+1\right](x^2-x)-(x^2+1)D_x\left[x^2-x\right]}{(x^2-x)^2} \\ &= \frac{(2x+0)(x^2-x) \;-\; (x^2+1)(2x-1)}{(x^2-x)^2} \end{aligned}$$

This is the derivative, but a few extra steps of algebra simplify our answer.

$$\begin{aligned} &= \frac{(2x^3-2x^2) \;-\; (2x^3-x^2+2x-1)}{(x^2-x)^2} \\ &= \boxed{\frac{-x^2-2x+1}{(x^2-x)^2}}. \end{aligned}$$

✍

Example 20.5 Find the derivative of $\dfrac{x^5}{e^x}$.

This function is a quotient, so we apply the quotient rule to find its derivative.

$$\begin{aligned} D_x\left[\frac{x^5}{e^x}\right] &= \frac{D_x\left[x^5\right]e^x - x^5 D_x\left[e^x\right]}{(e^x)^2} \\ &= \frac{5x^4e^x \;-\; x^5e^x}{e^xe^x}. \end{aligned}$$

This is the derivative, but some simplification is possible:

$$= \frac{x^4e^x(5-x)}{e^xe^x} = \boxed{\frac{x^4(5-x)}{e^x}}.$$

✍

Example 20.6 The quotient rule can be computationally expensive, so don't use it if you don't have to. As an example, consider differentiating $\frac{3x^2+4x-5}{2}$. This is a quotient, so you could use the quotient rule. But the denominator is constant, so a better choice would be to factor it out using the constant multiple rule:

$$D_x\left[\frac{3x^2+4x-5}{2}\right] = \frac{1}{2}\cdot D_x\left[3x^2+4x-5\right] = \frac{1}{2}(6x+4) = \boxed{3x+2}.$$

If you used the quotient rule, your work would go like this:

$$\begin{aligned} D_x\left[\frac{3x^2+4x-5}{2}\right] &= \frac{D_x\left[3x^2+4x-5\right]\cdot 2+\left(3x^2+4x-5\right)D_x\left[2\right]}{2^2} \\ &= \frac{(6x+4)\cdot 2+\left(3x^2+4x-5\right)\cdot 0}{4} \\ &= \frac{(6x+4)\cdot 2}{4} = \boxed{3x+2}. \end{aligned}$$

Though it gave the same answer, the quotient rule was an overkill. Work enough exercises that you see your choices and their consequences. ✍

Example 20.7 Find the derivative of x^{-10}.

We will do this two ways. First, we can simply apply the power rule.

$$D_x\left[x^{-10}\right] = -10x^{-10-1} = \boxed{-10x^{-11}}.$$

Our second step uses the quotient rule.

$$\begin{aligned} D_x\left[x^{-10}\right] = D_x\left[\frac{1}{x^{10}}\right] &= \frac{D_x\left[1\right]x^{10}-1\cdot D_x\left[x^{10}\right]}{\left(x^{10}\right)^2} \\ &= \frac{0\cdot x^{10}-10x^9}{x^{20}} = -10x^{9-20} = \boxed{-10x^{-11}}. \end{aligned}$$ ✍

Besides reminding us that a problem can be done several ways, this example raises an important point. In deriving the power rule $D_x[x^n] = nx^{n-1}$ in Chapter 17, we proved it *only for positive integer values of n*. But we asserted then that it actually holds for all real values of n, positive or negative, and that that we would see this in due time. Example 20.7 suggests that we can use the quotient rule to show that the power rule holds for *negative integer* values of n. Exercise 13 below asks for the details.

Example 20.8 Find all x for which the tangent to $y = xe^x$ at $(x, f(x))$ has slope 0.

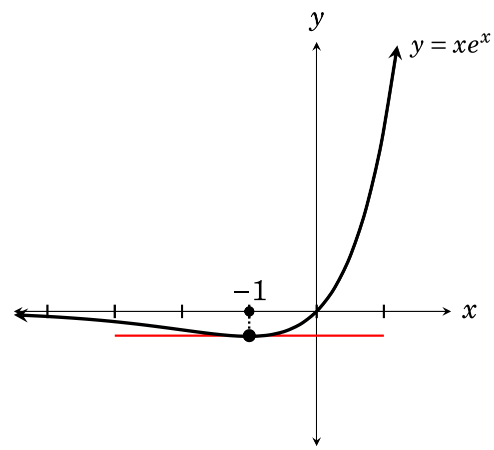
Solution: The slope of the tangent line is given by the derivative, so as in the previous example our first task is to find the derivative of $y = xe^x$.

This requires the product rule:

$$\frac{dy}{dx} = 1 \cdot e^x + x \cdot e^x = e^x(1+x).$$

This means the tangent has slope 0 where $e^x(1+x) = 0$. Because $e^x > 0$ for all x, the only way we can have $e^x(1+x) = 0$ is if $x = -1$.

Answer: The tangent to $y = xe^x$ has slope 0 at $x = -1$. (See the graph on the right.) ✍



Exercises for Chapter 20

In problems 1–12 find the indicated derivative.

1. $D_x\left[(2x^4+3x)(3x^2-x)\right]$

2. $D_x\left[e^x\sqrt{x}\right]$

3. $D_x\left[xe^x\right]$

4. $D_x\left[e^{2x}\right]$ (Hint: $e^{2x} = e^x e^x$.)

5. $D_x\left[\dfrac{x^2+x}{x+5}\right]$

6. $D_x\left[\dfrac{x^2+3x-4}{x+\sqrt{5}}\right]$

7. $D_x\left[\dfrac{e^x+x}{x^3+x^2+1}\right]$

8. $D_x\left[\dfrac{x^3+x^2+1}{x}\right]$

9. $D_x\left[\dfrac{x}{x^3+x^2+1}\right]$

10. $D_x\left[\dfrac{x+1}{x-1}\right]$

11. $D_x\left[\dfrac{1}{x^2+1}\right]$

12. $D_x\left[\dfrac{e^x}{\sqrt{x}}\right]$

13. In Chapter 17 we proved that the power rule $D_x[x^n] = nx^{n-1}$ works for positive integer values of n. Combine this fact with the quotient rule to show that the power rule also holds for negative integer values of n. (See Example 20.7.)

14. Find the equation of the tangent line to the graph of $f(x) = e^x\sqrt{x}$ at point $(1, f(1))$.

15. Find the equation of the tangent line to the graph of $f(x) = \dfrac{1}{\sqrt{x}}$ at point $(4, f(4))$.

16. Find all points (x, y) on the graph of $y = x + \dfrac{1}{x-3}$ where the tangent has slope 0.

17. Find all points (x, y) on the graph of $y = \frac{x}{e^x}$ where the tangent line is horizontal.

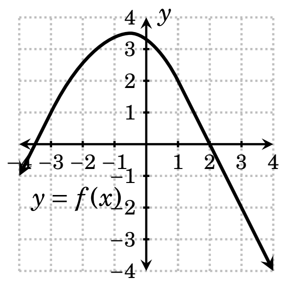
18. Two functions $f(x)$ and $g(x)$ are graphed below.

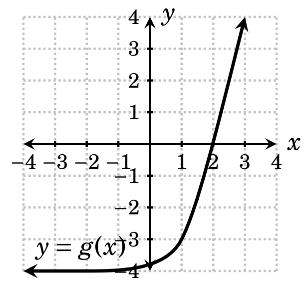
Estimate $h'(1.5)$ if:

(a) $h(x) = f(x)g(x)$.

(b) $h(x) = \frac{f(x)}{g(x)}$.

(c) $h(x) = \frac{g(x)}{f(x)}$.





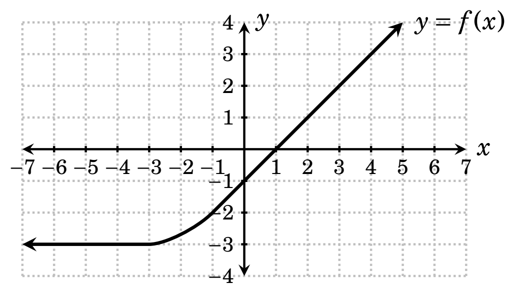
19. A function $f(x)$ is graphed below.

Find $g'(3)$ if:

(a) $g(x) = x^2 f(x)$.

(b) $g(x) = \frac{x^2}{f(x)}$.

(c) $g(x) = \left(f(x)\right)^2$.



20. Information about $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ is given in the table below.
Find $h'(3)$ if:

(a) $h(x) = \left(x + x^3\right) f(x)$.

(b) $h(x) = \frac{g(x)}{f(x)}$.

(c) $h(x) = f(x)g(x)$.

x	0	1	2	3	4	5
$f(x)$	−4	−2	0	1	1	0
$f'(x)$	2	1	1	3	0.5	−1
$g(x)$	10	9	7	4	0	−4
$g'(x)$	0	−0.5	−1	−3	−4	−4

Exercise Solutions for Chapter 20

1. $D_x\left[(2x^4+3x)(3x^2-x)\right]=(8x^3+3)(3x^2-x)+(2x^4+3x)(6x-1)$

3. $D_x\left[xe^x\right]=1\cdot e^x+xe^x=e^x+xe^x$

5. $D_x\left[\dfrac{x^2+x}{x+5}\right]=\dfrac{(2x+1)(x+5)-(x^2+x)(1+0)}{(x+5)^2}=\dfrac{2x^2+11x+5-x^2-x}{(x+5)^2}=\dfrac{x^2+10x+5}{(x+5)^2}$

7. $D_x\left[\dfrac{e^x+x}{x^3+x^2+1}\right]=\dfrac{(e^x+1)(x^3+x^2+1)-(e^x+x)(3x^2+2x)}{(x^3+x^2+1)^2}$

9. $D_x\left[\dfrac{x}{x^3+x^2+1}\right]=\dfrac{1\cdot(x^3+x^2+1)-x(3x^2+2x)}{(x^3+x^2+1)^2}=\dfrac{-2x^3-x^2+1}{(x^3+x^2+1)^2}$

11. $D_x\left[\dfrac{1}{x^2+1}\right]=\dfrac{0\cdot(x^2+1)-1\cdot(2x+0)}{(x^2+1)^2}=\dfrac{-2x}{(x^2+1)^2}$

13. In Chapter 17 we proved that the power rule $D_x[x^n]=nx^{n-1}$ works for positive integer values of n. Combine this fact with the quotient rule to show that the power rule also holds for negative integer values of n.

Let n be a negative integer, so $n=-m$ for a the *positive* integer m. Then the power rule as proved in Chapter 17 says $D_x[x^m]=mx^{m-1}$. This problem is asking us to show $D_x[x^n]=nx^{n-1}$. To confirm this with the quotient rule, observe that

$$D_x\left[x^n\right]=D_x\left[x^{-m}\right]=D_x\left[\frac{1}{x^m}\right]=\frac{0\cdot x^m-1\cdot mx^{m-1}}{(x^m)^2}=\frac{-mx^{m-1}}{x^{2m}}=-mx^{-m-1}=nx^{n-1}.$$

15. Find the equation of the tangent line to the graph of $f(x)=\dfrac{1}{\sqrt{x}}$ at point $(4,f(4))$.

The derivative $f'(x)$ will tell us the slope of this line. Finding $f'(x)$ will be easier with the product rule than the quotient rule. Since $f(x)=x^{-1/2}$, we get $f'(x)=-\dfrac{1}{2}x^{-1/2-1}=-\dfrac{1}{2}x^{-3/2}=-\dfrac{1}{2x^{3/2}}=-\dfrac{1}{2\sqrt{x}^3}$. When $x=4$, the tangent line has slope is thus $m=f'(4)=-\dfrac{1}{2\sqrt{4}^3}=-\dfrac{1}{16}$, and passes through the point $(4,f(4))=(4,1/2)$. Now use the point-slope formula for the equation of a line.

$$\begin{aligned}
y-y_0 &= m(x-x_0)\\
y-\frac{1}{2} &= -\frac{1}{16}(x-4)\\
y-\frac{1}{2} &= -\frac{1}{16}x+\frac{1}{4}\\
\textbf{Answer:}\quad y &= -\frac{1}{16}x+\frac{3}{4}
\end{aligned}$$

17. Find all points (x, y) on the graph of $y = \dfrac{x}{e^x}$ where the tangent line is horizontal.

We need to find where the tangent line to has zero slope, that is, where $\frac{dy}{dx} = 0$. By the quotient rule, $\dfrac{dy}{dx} = \dfrac{1 \cdot e^x - xe^x}{(e^x)^2} = \dfrac{e^x(x-1)}{(e^x)^2} = \dfrac{x-1}{e^x}$. From this we can see that the only way the derivative can be 0 is if $x = 1$. Therefore the tangent line to $y = \frac{x}{e^x}$ is horizontal only at the point $\left(1, \frac{1}{e^1}\right) = (1, 1/e)$.

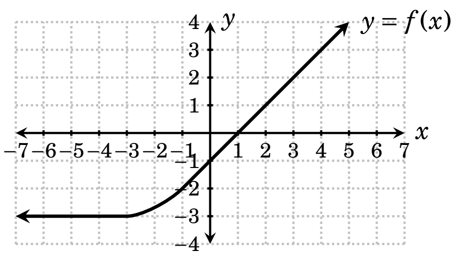
19. A function $f(x)$ is graphed below.

Find $g'(3)$ if:

(a) $g(x) = x^2 f(x)$.

(b) $g(x) = \dfrac{x^2}{f(x)}$.

(c) $g(x) = \big(f(x)\big)^2$.



From the graph, $g(3) = 2$ and $g'(3) = 1$. Now use the product and quotient rules:

(a) Note $g'(x) = 2xf(x) + x^2 g'(x)$, so $g'(3) = 2 \cdot 3 \cdot f(3) + 3^2 g'(3) = 2 \cdot 3 \cdot 2 + 9 \cdot 1 = \boxed{21}$

(b) Note $g'(x) = \frac{2xf(x) - x^2 f'(x)}{(f(x))^2}$, so $g'(3) = \frac{2 \cdot 3 \cdot f(3) - 3^2 f'(3)}{(f(3))^2} = \frac{2 \cdot 3 \cdot 2 - 3^2 \cdot 1}{2^2} = \boxed{\frac{3}{4}}$

(c) Write $g(x) = f(x)f(x)$, so by the product rule $g'(x) = f'(x)f(x) + f(x)f'(x) = 2f(x)f'(x)$. Therefore $g'(3) = 2f(3)f'(3) = 2 \cdot 2 \cdot 1 = \boxed{4}$.