CHAPTER 22

Higher Derivatives

Consider a function, say $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$. Take its derivative.

$$f'(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1$$

The derivative is a function too, and we can take *its* derivative. We call this the *second derivative* of f, and denote it as f''. The derivative of the second derivative is called the *third derivative* of f, and so on.

$$\begin{aligned}
f''(x) &= 20x^3 + 12x^2 + 6x + 2 && \text{(2nd derivative)} \\
f'''(x) &= 60x^2 + 24x + 6 && \text{(3rd derivative)} \\
f^{(4)}(x) &= 120x + 24 && \text{(4th derivative)} \\
f^{(5)}(x) &= 120 && \text{(5th derivative)} \\
f^{(6)}(x) &= 0 && \text{(6th derivative)} \\
f^{(7)}(x) &= 0 && \text{(7th derivative)} \\
&\vdots && \vdots
\end{aligned}$$

By convention, the fourth derivative is denoted $f^{(4)}$ instead of f''''. The fifth derivative is denoted $f^{(5)}$ instead of f''''', and so on. The derivatives of the derivatives of a function are called it's **higher derivatives**.

In the example above the higher derivatives are eventually all zero, but that needn't always be the case. Consider $g(x) = x^{-1}$.

$$\begin{aligned}
g(x) &= x^{-1} && \\
g'(x) &= -x^{-2} && \text{(1st derivative)} \\
g''(x) &= 2x^{-3} && \text{(2nd derivative)} \\
g'''(x) &= -6x^{-4} && \text{(3rd derivative)} \\
g^{(4)}(x) &= 24x^{-5} && \text{(4th derivative)} \\
g^{(5)}(x) &= -120x^{-6} && \text{(5th derivative)}
\end{aligned}$$

The derivatives of g get more and more complex the higher you go.

The higher derivatives of $\sin(x)$ repeat every fourth derivative:

$$\begin{aligned}
y &= \sin(x) & \\
y' &= \cos(x) & \text{(1st derivative)} \\
y'' &= -\sin(x) & \text{(2nd derivative)} \\
y''' &= -\cos(x) & \text{(3rd derivative)} \\
y^{(4)} &= \sin(x) & \text{(4th derivative)} \\
y^{(5)} &= \cos(x) & \text{(5th derivative)} \\
y^{(6)} &= -\sin(x) & \text{(6th derivative)} \\
y^{(7)} &= -\cos(x) & \text{(7th derivative)}
\end{aligned}$$

The higher derivatives of e^x are all the same:

$$\begin{aligned}
y &= e^x & \\
y' &= e^x & \text{(1st derivative)} \\
y'' &= e^x & \text{(2nd derivative)} \\
y''' &= e^x & \text{(3rd derivative)}
\end{aligned}$$

As suggested above, the nth derivative of a function $y = f(x)$ can be denoted $y^{(n)}$ instead of $f^{(n)}(x)$.

There are other variations on notation. For example, the third derivative of a function $y = f(x)$ is $\frac{d}{dx}\left[\frac{d}{dx}\left[\frac{d}{dx}[y]\right]\right]$, which suggests the notation

$$\frac{d}{dx}\left[\frac{d}{dx}\left[\frac{d}{dx}[y]\right]\right] = \frac{d^3y}{dx^3}.$$

In this way, the nth derivative of $y = f(x)$ is sometimes denoted as $\dfrac{d^ny}{dx^n}$. Here are some different notations for higher derivatives of a function $y = f(x)$.

1st derivative:	$f'(x)$	y'	$D_x[f(x)]$	$\frac{dy}{dx}$	$\frac{d}{dx}\left[y\right]$	$\frac{d}{dx}\left[f(x)\right]$
2nd derivative:	$f''(x)$	y''	$D_x^2[f(x)]$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}\left[y\right]$	$\frac{d^2}{dx^2}\left[f(x)\right]$
3rd derivative:	$f'''(x)$	y'''	$D_x^3[f(x)]$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}\left[y\right]$	$\frac{d^3}{dx^3}\left[f(x)\right]$
4th derivative:	$f^{(4)}(x)$	$y^{(4)}$	$D_x^4[f(x)]$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}\left[y\right]$	$\frac{d^4}{dx^4}\left[f(x)\right]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Obviously these notations are sensitive to the variables at play. For example, the higher derivatives of a function $z = g(w)$ are denoted as follows.

1st derivative:	$g'(w)$	z'	$D_w\big[g(w)\big]$	$\dfrac{dz}{dw}$	$\dfrac{d}{dw}\Big[z\Big]$	$\dfrac{d}{dw}\Big[f(w)\Big]$
2nd derivative:	$g''(w)$	z''	$D_w^2\big[g(w)\big]$	$\dfrac{d^2z}{dw^2}$	$\dfrac{d^2}{dw^2}\Big[z\Big]$	$\dfrac{d^2}{dw^2}\Big[f(w)\Big]$
3rd derivative:	$g'''(w)$	z'''	$D_w^3\big[g(w)\big]$	$\dfrac{d^3z}{dw^3}$	$\dfrac{d^3}{dw^3}\Big[z\Big]$	$\dfrac{d^3}{dw^3}\Big[f(w)\Big]$
4th derivative:	$g^{(4)}(w)$	$z^{(4)}$	$D_w^4\big[g(w)\big]$	$\dfrac{d^4z}{dw^4}$	$\dfrac{d^4}{dw^4}\Big[z\Big]$	$\dfrac{d^4}{dw^4}\Big[f(w)\Big]$

Example 22.1 Find the first four derivatives of $y = xe^x$.

The function xe^x is a product, so we use the product rule for its derivative.

$$\frac{dy}{dx} = 1\cdot e^x + xe^x = (1+x)e^x$$

This is also a product, so again we use the product rule for its derivative.

$$\frac{d^2y}{dx^2} = 1\cdot e^x + (1+x)e^x = (2+x)e^x$$

Once again we have a product, so we continue with the product rule.

$$\frac{d^3y}{dx^3} = 1\cdot e^x + (2+x)e^x = (3+x)e^x$$

$$\frac{d^4y}{dx^4} = 1\cdot e^x + (3+x)e^x = (4+x)e^x$$

This pattern suggests that the nth derivative of xe^x is $\dfrac{d^ny}{dx^n} = (n+x)e^x$. ✍

Exercises for Chapter 22

Find the first, second, third and fourth derivatives of these functions.

1. $3x^8 - x^3$ **2.** $e^x\cos(x)$ **3.** $\sin(x)+\cos(x)$ **4.** $x\sin(x)$

5. $\sqrt{x}$ **6.** $\tan(x)$ **7.** x^2e^x **8.** $xe^x - x$

Exercise Solutions for Chapter 22

1. $y = 3x^8 - x^3$

$$y' = 24x^7 - 3x^2$$
$$y'' = 168x^6 - 6x$$
$$y''' = 1008x^5 - 6$$
$$y^{(4)} = 5040x^4$$

3. $y = \sin(x) + \cos(x)$

$$y' = \cos(x) - \sin(x)$$
$$y'' = -\sin(x) - \cos(x)$$
$$y''' = -\cos(x) + \sin(x)$$
$$y^{(4)} = \sin(x) + \cos(x)$$

5. $y = \sqrt{x} = x^{1/2}$

$$\begin{aligned} y' &= \tfrac{1}{2}x^{-1/2} &&= \frac{1}{2\sqrt{x}} \\ y'' &= -\tfrac{1}{4}x^{-3/2} &&= \frac{-1}{4\sqrt{x}^3} \\ y''' &= \tfrac{3}{8}x^{-5/2} &&= \frac{3}{8\sqrt{x}^5} \\ y^{(4)} &= -\tfrac{15}{16}x^{-7/2} &&= \frac{-15}{16\sqrt{x}^7} \end{aligned}$$

7. $y = x^2e^x$

$$\begin{aligned} y' &= 2xe^x + x^2e^x &&= (2x + x^2)e^x \\ y'' &= (2+2x)e^x + (2x + x^2)e^x &&= (2 + 4x + x^2)e^x \\ y''' &= (4+2x)e^x + (2 + 4x + x^2)e^x &&= (6 + 6x + x^2)e^x \\ y^{(4)} &= (6+2x)e^x + (6 + 6x + x^2)e^x &&= (12 + 8x + x^2)e^x \end{aligned}$$