# Related Rates

This chapter concerns a type of derivative problem in which we solve for one rate of change, given that we know some other rate of change. The technique hinges on implicit differentiation and the rate-of-change interpretation of the derivative.

For the most part we will deal with functions of time t. Recall from Chapter 26 that if $f(t)$ is a quantity that depends on time t, then $f'(t)$ gives the rate of change of the quantity $f(t)$ at time t.

Examples:

1. Suppose that $P(t)$ models the population of the world t years after the year 2010. Then $P'(t)$ equals the rate of change (in people/year) of the world's population at time t.

2. Suppose $w(t)$ equals the amount of water (gallons) in a tank at time t (minutes). Imagine that water is being added to the tank at a varying rate. Then $w'(t)$ equals the rate (in gallons/min) that water is being added or removed at time t.
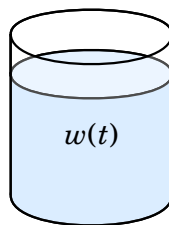

   For instance, if t is the number of minutes after noon, then $w'(125) = 2.37$ means that at 2:05 p.m. water is being added to the tank at a rate of 2.37 gallons per minute.

3. If an airplane has altitude h (meters) at time t (minutes), then $\frac{dh}{dt}$ gives the rate of change in altitude (in meters/minute).

Next let's look at a very simple example of the kinds of problems we will be solving in this chapter.

A Simple Related Rates Problem   Imagine that a perfectly spherical balloon is being inflated. It so happens that at the instant the radius r is 15 inches, r is increasing at a rate of 0.5 inches per minute.
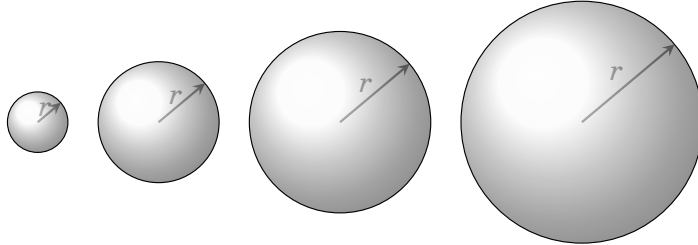
Question: When $r = 15$, how fast is the balloon's surface area S increasing?

As the balloon inflates, the radius r and surface area S are changing as time t increases. So both r and S are functions of time: $r = f(t)$ and $S = g(t)$ (although these functions are not given in the statement or the problem). It is stated that when radius r is 15 inches, it is increasing at a rate of 0.5 inches per minute. Therefore $\boxed{\text{we know } \frac{dr}{dt} = 0.5 \text{ when } r = 15.}$ The question is: $\boxed{\text{What is } \frac{dS}{dt} \text{ when } r = 15?}$

So in this problem *we know one rate of change and we want to find another.* The key to solving such a problem is to find an equation relating r and S. In this case we can use the formula for the surface area of a sphere:

$$S = 4\pi r^2.$$

Don't forget: S and r are really functions of time t, so think of this equation as saying $g(t) = 4\pi\big(f(t)\big)^2$. Now differentiate both sides with respect to t.

$$\frac{d}{dt}\Big[S\Big] = \frac{d}{dt}\Big[4\pi r^2\Big]$$
$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

Suddenly we have an equation relating the two rates $\frac{dS}{dt}$ and $\frac{dr}{dt}$. And we know $\frac{dr}{dt} = 0.5$ when $r = 15$. So for $r = 15$, the above equation gives

$$\frac{dS}{dt} = 8\pi \cdot 15 \cdot 0.5 = 60\pi.$$

Answer: When $r = 15$, the balloon's surface area is increasing at a rate of 60π square inches per minute.

The kind of problem we just solved is called a **related rates problem.** In such a problem, one (or more) rates of change is known, and another needs to be found. There is a standard procedure for solving related rates problems, and it mirrors the steps we just took above in our first example.

To solve a related rates problem:

1. Analyze the problem. Draw a diagram if necessary.

2. Identify the quantity whose rate of change you know.
   Identify the quantity whose rate of change you need to find.

3. Find an equation relating these quantities.

4. Differentiate both sides of the equation with respect to time t
   (or with respect to the variable the quantities depend upon).
   You will get an equation involving the relevant rates.

5. Plug in any constants.

6. Solve for the unknown rate. (The order of steps 5 and 6 can be reversed.)

Example 29.1 This problem concerns a conical tank that is 4 meters high and has a radius of 2 meters, as shown below. Water pours into the tank at a constant rate of 0.25 cubic meters per minute. As this happens the volume V of water in the tank increases, and the water level h rises.
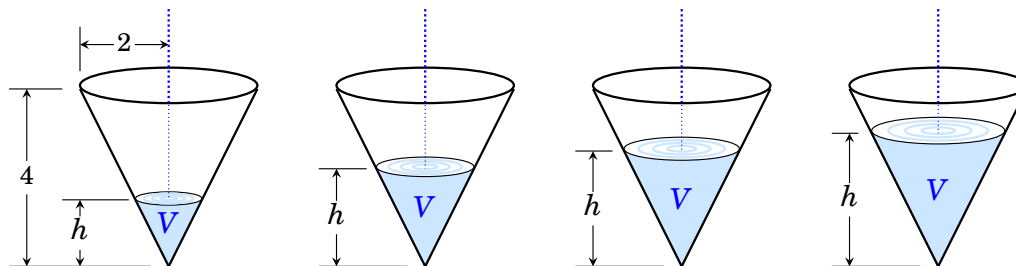
Question: How quickly is the water level h rising at the precise instant that $h = 2$ meters?

To answer this question first note that both V and h change with time, so they are functions of time t. We might underscore this by writing

$$V = f(t) \quad \text{and} \quad h = g(t).$$

It is stated that volume is changing at a rate of 0.25 cubic meters per minute. And we are asked to find the rate of change of h. Let's summarize this as

$$\text{Know: } \frac{dV}{dt} = 0.25 \qquad \text{Want: } \frac{dh}{dt} \text{ (when } h = 2\text{).}$$

The next step is to find an equation relating V and h.
The water in the tank has the shape of a cone of
height h and some radius r. So we can use the
formula for the volume of a cone, which is

$$V = \frac{1}{3}\pi r^2 h.$$

But this equation has *three* variables V, r and h. We
need an equation relating just V and h. By similar
triangles, $\frac{r}{h} = \frac{2}{4}$, so $r = \frac{h}{2}$, and hence $V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$.
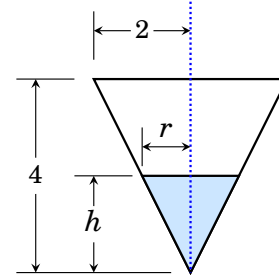
Therefore volume V and water level h are related by the equation

$$V = \frac{\pi}{12}h^3.$$

The next step is to differentiate implicitly with respect to t.

$$\frac{d}{dt}\Big[V\Big] \quad = \quad \frac{d}{dt}\Big[\frac{\pi}{12}h^3\Big]$$

$$\frac{dV}{dt} \quad = \quad \frac{\pi}{12}3h^2\frac{dh}{dt}$$

Recall that we know $\frac{dV}{dt} = 0.25$ and we are looking for $\frac{dh}{dt}$. The above becomes

$$0.25 \quad = \quad \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Solving for the rate we want,

$$\frac{dh}{dt} \quad = \quad \frac{1}{\pi h^2}. \tag{*}$$

Actually, we seek $\frac{dh}{dt}$ at the precise instant that $h = 2$, and this is

$$\frac{dh}{dt} \quad = \quad \frac{1}{\pi 2^2} = \frac{1}{4\pi}.$$

Answer: At the instant $h = 2$, the water level h is rising at a rate of
$\frac{1}{4\pi} \approx 0.795$ meters/min.

Notice how in Equation (*), the rate $\frac{dh}{dt}$ gets smaller as h gets bigger.
This makes sense, because as h gets bigger the water's circular surface gets
bigger too. The added water has to spread out over a bigger surface, so it
takes more water to raise the level one centimeter than it does for smaller h.

Example 29.2 A plane is taxiing down a runway that is one mile from a tower, as shown below (left). When the plane is 5/3 miles from the tower, the distance y between tower and plane is increasing at a rate of 100 mph. What is the plane's velocity at this point in time?

$x = f(t)$
$y = g(t)$
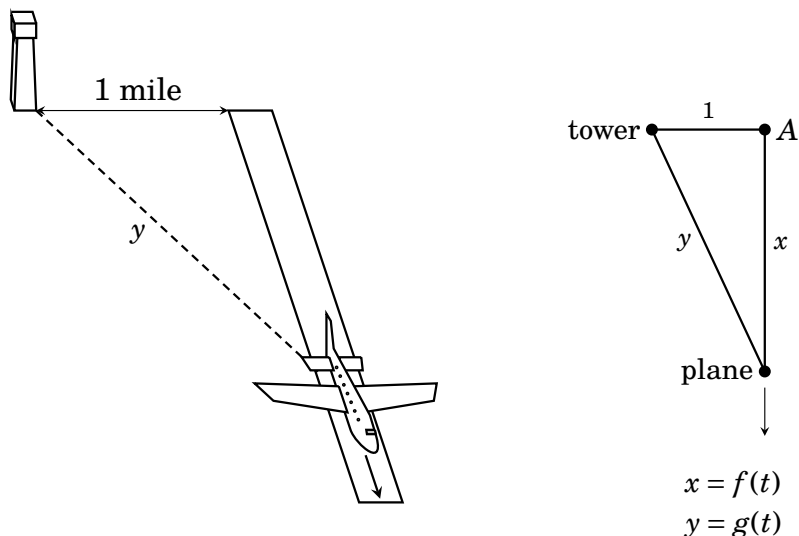
Let's begin by analyzing this situation with a diagram. The tower, plane and runway form a right triangle as indicated above (right). Let the distance from plane to tower be y, and the distance from plane to point A be x. This is a dynamic picture, as both x and y change with time t. In the background we have in mind that $x = f(t)$ and $y = g(t)$.

So $x = f(t)$ is a position function for the plane, and thus the plane's velocity is given by $\frac{dx}{dt}$, while $\frac{dy}{dt}$ gives the rate of change of the distance from plane to tower. Let's take stock of the rate we know and the rate we want to find.

$$\text{Know: } \frac{dy}{dt} = 100 \qquad (\text{when } y = \tfrac{5}{3})$$

$$\text{Want: } \frac{dx}{dt} \qquad\qquad (\text{when } y = \tfrac{5}{3}).$$

Since these are rates of change of x and y, the next step is to find an equation relating x and y. The Pythagorean theorem gives one:

$$x^2 + 1^2 = y^2.$$

Next we differentiate with respect of t to bring $\frac{dx}{dt}$ and $\frac{dy}{dt}$ into the picture.

$$\frac{d}{dt}\left[x^2+1\right] = \frac{d}{dt}\left[y^2\right]$$
$$2x\frac{dx}{dt}+0 = 2y\frac{dy}{dt}$$

We are looking for $\frac{dx}{dt}$, so solve for it.

$$\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$$

This gives the plane's velocity $\frac{dx}{dt}$ in terms of x, y and $\frac{dy}{dt}$. We seek $\frac{dx}{dt}$ at the instant that $y=\frac{5}{3}$ mile. We know that at this instant $\frac{dy}{dt}=100$. Thus

$$\frac{dx}{dt} = \frac{5/3}{x}\cdot 100. \qquad (*)$$

But what value should the x get? To find out, we can go back to our triangle, drawn below for $y=\frac{5}{3}$. The Pythagorean theorem gives

$$\begin{aligned} x^2+1^2 &= \left(\frac{5}{3}\right)^2 \\ x^2 &= \frac{25}{9}-1 \\ x^2 &= \frac{16}{9} \\ x &= \frac{4}{3}. \end{aligned}$$
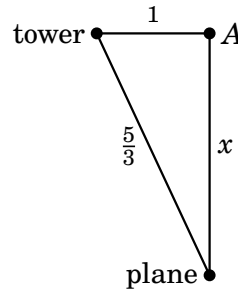

Finally we can put $x=4/3$ into the above equation (*) to get

$$\frac{dx}{dt} = \frac{5/3}{4/3}\cdot 100 = 125\,\text{mph}.$$

Answer: When 5/3 miles from the tower, the plane's velocity is 125 mph.✍

It is an annoying fact that, in an introductory calculus course, many of the related rates problems seem like artificial academic exercises. This is both true and necessary. Most serious real-world problems are a little too complex for us at this early stage of the game. But doing related rates problems moves us ever closer to meaningful applications of calculus.

Example 29.3 A six-foot-tall person walks away from an 18-foot-tall street-light, along a level sidewalk at night, at a rate of five feet per second. How fast is the tip A of the shadow moving down the sidewalk?
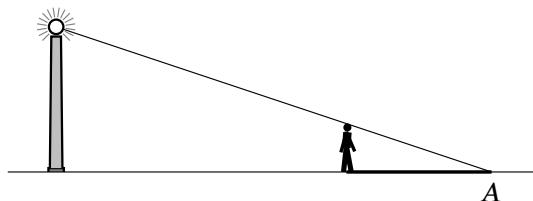
Solution   Represent this situation visually with the following diagram, where segment DE is the pole and BC is the person. Let y be the distance from pole to person, and let x be the distance from pole to shadow tip.
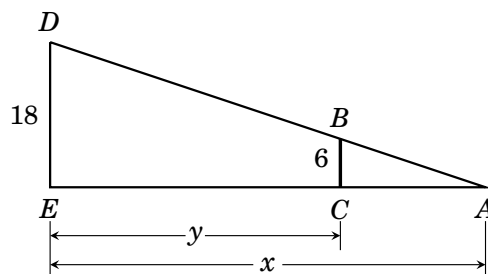
Then y is a position function for the person, and x is a position function for the shadow's tip. Here's what we know and what we want:

$$\text{We know: } \frac{dy}{dt} = 5 \text{ feet per second}$$

$$\text{We want: } \frac{dx}{dt}.$$

We can relate the variable x and y by similar triangles. Triangle ADE is similar to ABC, so the ratios of vertical-to-horizontal sides of these two triangles are equal. Thus

$$\frac{x}{18} = \frac{x-y}{6}$$

$$x = 3(x-y)$$

$$x = \frac{3}{2}y$$

In words, however far the person is from the pole, the shadow's tip is always three halves that distance from the pole. Differentiating both sides,

$$\frac{d}{dt}\left[x\right] = \frac{d}{dt}\left[\frac{3}{2}y\right]$$

$$\frac{dx}{dt} = \frac{3}{2}\frac{dy}{dt}. \tag{*}$$

(This says that your shadow tip moves away from you at 3/2 the speed you are walking.) Combining (*) with $\frac{dy}{dt} = 5$ gives us the shadow tip's velocity:

$$\frac{dx}{dt} = \frac{3}{2}\cdot 5 = \frac{15}{2} \text{ feet per second.}$$

Answer   The shadow's tip moves at a constant velocity of 7.5 feet per second.   ✐

Exercises for Chapter 29

1. A spherical balloon is inflated at a rate of 100π cubic feet per minute. How fast is the radius increasing at the instant that the radius is 5 feet?
(A sphere of radius r has volume is $V = \frac{4}{3}\pi r^3$)

2. A spherical balloon is inflated at a rate of 100π cubic feet per minute. In Exercise 1 above you found that when its radius is 5 feet, the radius is increasing at a rate of one foot per minute. Now find how fast the surface area is increasing at the instant that the radius is 5 feet. (A sphere of radius r has surface area is $S = 4\pi r^2$.)

3. A 13-foot ladder is leaning against a wall, as illustrated (right), when its base begins to slide away from the wall at a rate of 5 feet per second. How fast is the top of the ladder sliding down the wall when the base of the ladder is 12 feet from the wall?

$5\,\frac{ft}{sec}$

4. A 13-foot ladder is leaning against a wall, as illustrated (right), when its base begins to slide away from the wall at a rate of 5 feet per second. At what rate is the angle θ changing when the base is 12 feet from the wall?
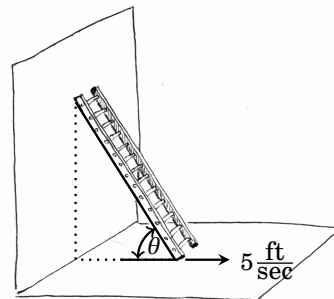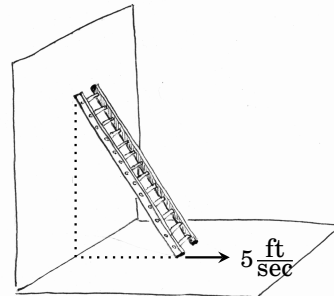

$5\,\frac{ft}{sec}$

5. Water flows into a conical tank (see illustration) at a rate of 9 cubic feet per minute. How fast is the water level h rising when the water is 6 feet deep? (The volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)
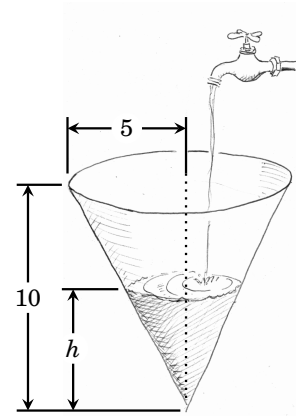
6. A spherical balloon is inflated and its volume increases at a rate of 15 cubic inches per minute. What is the rate of change of its radius when the radius is 10 inches?

7. Sand falls at a rate of 6 cubic feet per minute, making a conical pile whose height h is always half its radius r. Find the rate of change of the radius r (in feet/min) when $r = 2$ feet. (The volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)
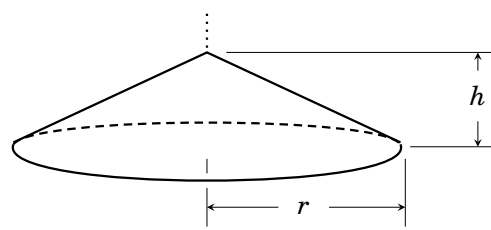
8. A stone is dropped into a lake, causing circular waves whose radii increase at a rate of 2 meters per second. At what rate is the circumference of a wave changing when its radius is 5 meters?

Exercise Solutions for Chapter 29

1. A spherical balloon is inflated at a rate of 100π cubic feet per minute. How fast is the radius increasing at the instant that the radius is 5 feet?
(A sphere of radius r has volume is $V = \frac{4}{3}\pi r^3$)

Let V be the volume of the sphere and let r be its radius. In this problem we know $\frac{dV}{dt} = 100\pi$ cubic feet per minute, and we want to find $\frac{dr}{dt}$. To relate these two rates, start with the volume formula $V = \frac{4}{3}\pi r^3$.

$$\frac{d}{dt}\Big[V\Big] \;=\; \frac{d}{dt}\Big[\frac{4}{3}\pi r^3\Big] \qquad\qquad\text{(differentiate)}$$

$$\frac{dV}{dt} \;=\; 4\pi r^2\frac{dr}{dt}$$

$$100\pi \;=\; 4\pi 5^2\frac{dr}{dt} \qquad\qquad (r = 5 \text{ and } \tfrac{dV}{dt} = 100\pi)$$

Answer $\quad\dfrac{dr}{dt} \;=\; 1$ foot per minute

3. A 13-foot ladder is leaning against a wall, as illustrated below (left), when its base begins to slide away from the wall at a rate of 5 feet per second. How fast is the top of the ladder sliding down the wall when the base of the ladder is 12 feet from the wall?

The ladder, wall and floor form a right triangle whose hypotenuse has length 13, as illustrated. Say the horizontal side has length x and the vertical side has length y, so x and y are functions of time t. We can regard x as a position function for the base of the ladder and y as a position function for the top of the ladder. Thus $\frac{dx}{dt}$ is the velocity of the ladder's base, and $\frac{dy}{dt}$ is the velocity of the top ladder's top.
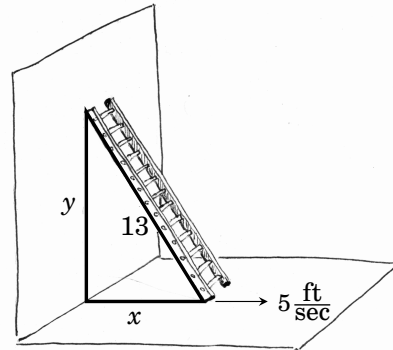
We know $\frac{dx}{dt} = 5$ (feet per second) and we seek $\frac{dy}{dt}$ at the instant that $x = 12$. The Pythagorean theorem gives an equation relating x and y, namely $x^2 + y^2 = 13^2$.
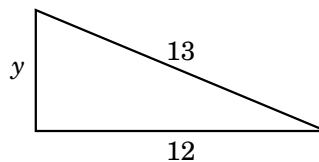
$$\frac{d}{dt}\Big[x^2 + y^2\Big] \;=\; \frac{d}{dt}\Big[13^2\Big]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} \;=\; 0$$

$$2\cdot 12\cdot 5 + 2y\frac{dy}{dt} \;=\; 0$$

$$\frac{dy}{dt} \;=\; -\frac{60}{y} \qquad\qquad (*)$$

To get an answer we just need to find the value for y and put it into Equation (*). When $x = 12$ the triangle is as shown on the right. The Pythagorean theorem then yields $y = 5$.

Answer When the base of the ladder is 12 feet from the wall, the top is sliding down the wall with a velocity of $\frac{dy}{dt} = -\frac{60}{5} = -12$ feet per second.

5. Water flows into a conical tank (see illustration) at a rate of 9 cubic feet per minute. How fast is the water level h rising when the water is 6 feet deep? (The volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)

The water in the tank forms a cone of volume V and height h, and both are functions of time t. We know that $\frac{dV}{dt} = 9$ cubic feet per second, and we need to find $\frac{dh}{dt}$ at the instant that $h = 6$.

The volume formula $V = \frac{1}{3}\pi r^2 h$ relates V and h, but it has a third variable r. We can get r in terms of h by similar triangles. By the diagram, $\frac{5}{10} = \frac{r}{h}$, so $r = \frac{h}{2}$. Plug this into the volume formula:

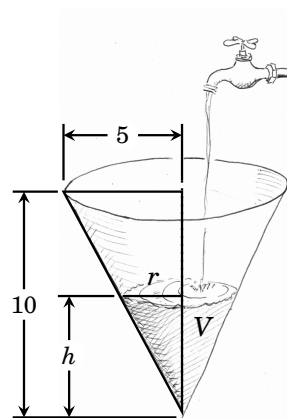$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}.$$

Next we differentiate both sides of this equation implicitly with respect to t.

$$\frac{d}{dt}\left[V\right] = \frac{d}{dt}\left[\frac{\pi h^3}{12}\right]$$

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$

We know $\frac{dV}{dt} = 9$, and we are interested in the instant that $h = 6$, so this becomes

$$9 = \frac{\pi 6^2}{4}\frac{dh}{dt}$$

$$9 = 9\pi\frac{dh}{dt}$$

Answer $\frac{dh}{dt} = \frac{1}{\pi}$ feet per minute

7. Sand falls at a rate of 6 cubic feet per minute, making a conical pile whose height h is always half its radius r. Find the rate of change of the radius r (in feet/min) when $r = 2$ feet.

The problems states that $\frac{dV}{dt} = 6$ and $h = \frac{1}{2}r$. Given this, we wish to find $\frac{dr}{dt}$ when $r = 2$. Start with the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$ and insert

$h = \frac{1}{2}r$ to get $V = \frac{1}{6}\pi r^3$. Next differentiate both sides of this with respect to t.

$$D_t[V] = D_t\left[\frac{1}{6}\pi r^3\right]$$

$$\frac{dV}{dt} = \frac{1}{2}\pi r^2 \frac{dr}{dt}$$

$$6 = \frac{1}{2}\pi 2^2 \frac{dr}{dt} \qquad\qquad \text{(insert } dV/dt = 6 \text{ and } r = 2\text{)}$$

$$\frac{dr}{dt} = \frac{3}{\pi} \text{ feet per minute}$$