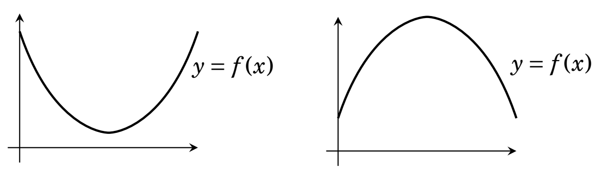
CHAPTER 32

Concavity and the Second Derivative Test

Given a function f, we've learned that its derivative f' tells us something about the shape of the graph of f, namely where it is increasing and decreasing. The function f increases where f' is positive and decreases where f' is negative.

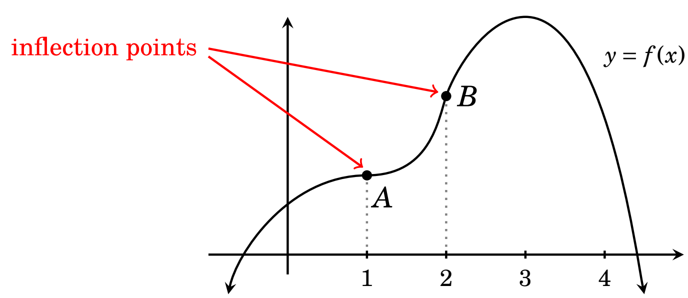
This chapter investigates what the *second derivative* f'' tells us about the shape of graph of $f(x)$. As we will see, f'' gives information about the *concavity* of the graph of f.

But first, a few informal definitions and ideas. A curve is **concave up** if it has the shape of a bowl that would hold water. It is **concave down** if it has the shape of an upside down bowl. This is illustrated below.



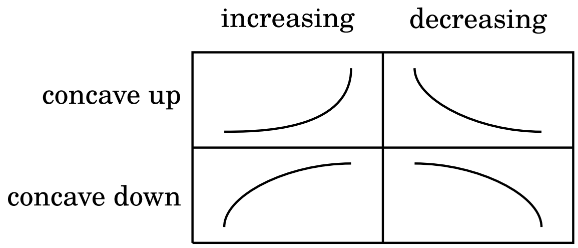
concave up **concave down**

The graph of a function can be concave up on some intervals and concave down on others. The graph shown below is concave down on the intervals $(-\infty, 1)$ and $(2, \infty)$. It is concave up on $(1, 2)$.



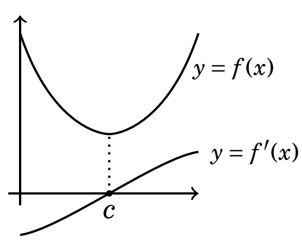
A point on the graph at which the concavity changes is called an **inflection point**. The graph above has two inflection points A (where the concavity changes from down to up), and B (where the concavity changes from up to down).

Note that a graph (or a portion of a graph) that is concave up (or down) can be increasing or decreasing. The four possibilities are shown in the chart below.



Let's now investigate how concavity is determined by the sign of the second derivative. We'll consider the concave up and down situations side-by-side and record our conclusion at the bottom of the page.

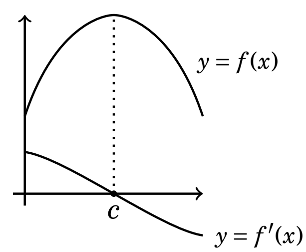
First consider the concave up graph of a function $y = f(x)$ shown below. The tangents to $y = f(x)$ have negative slope to the left of c, but positive slopes to the right of c. As x moves past c the tangent slope at $(x, f(x))$ increases, changing from negative to positive.



Therefore the derivative f' *increases*. Since the derivative of an increasing function is positive, $f''(x) > 0$.

Conclusion: A function is concave up wherever its second derivative is positive.

Now consider the concave down graph of a function $y = f(x)$ shown below. The tangents to $y = f(x)$ have positive slope to the left of c, and negative slopes the right of c. As x moves past c the tangent slope at $(x, f(x))$ decreases, changing from positive to negative.



Thus the derivative f' *decreases*. As the derivative of a decreasing function is negative, $f''(x) < 0$.

Conclusion: A function is concave down where its second derivative is negative.

Fact 32.1 Concavity

A function $f(x)$ is concave up on an interval if $f''(x) > 0$ on the interval.
A function $f(x)$ is concave down on an interval if $f''(x) < 0$ on the interval.

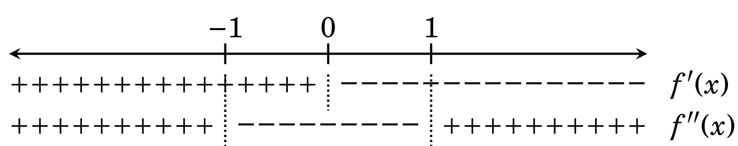
Example 32.1 Consider the function $f(x) = \dfrac{6}{x^2+3}$. Find the intervals on which this function is increasing, decreasing, concave up and concave down. Find all extrema, inflection points, and sketch the graph.

Solution: By the quotient rule, the derivative is $f'(x) = \dfrac{-12x}{\left(x^2+3\right)^2}$. The denominator is positive for any x (because it is squared), so the sign of $f'(x)$ is the same as the sign of its numerator $-12x$. Thus $f'(x)$ is positive when x is negative and negative when x is positive. We tally this in the chart below. By the first derivative test, the point $(0, f(0)) = (0,2)$ is a local maximum.

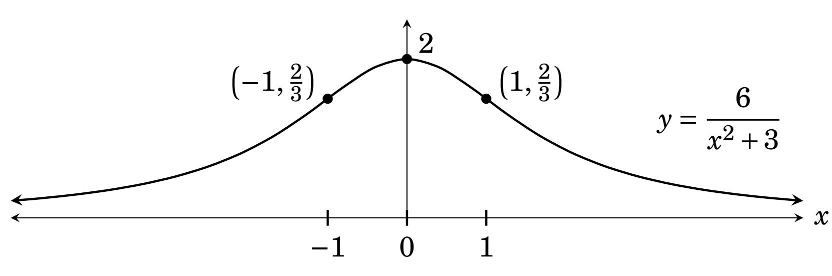
To find the concavity we investigate the second derivative, which is

$$\begin{aligned} f''(x) &= \frac{-12\left(x^2+3\right)^2 - (-12x)2\left(x^2+3\right)2x}{\left(x^2+3\right)^4} = \frac{-12\left(x^2+3\right)\left(\left(x^2+3\right)-4x^2\right)}{\left(x^2+3\right)^4} \\ &= \frac{-12\left(3-3x^2\right)}{\left(x^2+3\right)^3} = \frac{-36\left(1-x^2\right)}{\left(x^2+3\right)^3} = \frac{-36(1+x)(1-x)}{\left(x^2+3\right)^3}. \end{aligned}$$

The denominator is positive for all x, so the sign of $f''(x)$ is the sign of its numerator. The factored numerator tells us $f''(x)$ is zero if $x = \pm 1$, and that $f''(x)$ is positive on $(-\infty,-1)$ and $(1,\infty)$, and negative on $(-1,1)$. We tally this information in the chart below.

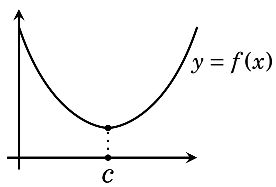


The concavity changes at 1 and 1, so the inflection points are $\left(-1, f(-1)\right) = \left(-1, \frac{2}{3}\right)$ and $\left(1, f(1)\right) = \left(1, \frac{2}{3}\right)$. From the chart, f is increasing and concave up on $(-\infty,-1)$. It is increasing and concave down on $(-1,0)$, and decreasing and concave down on $(0,1)$. On $(1,\infty)$, f is decreasing and concave up.



Note also that $f(x)$ is positive for all x, and $\lim\limits_{x\to\pm\infty} f(x) = 0$, so the line $y=0$ is a horizontal asymptote. All of this information yields the graph above. ✍

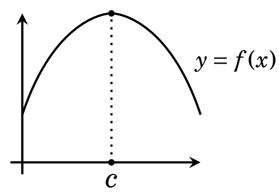
There is an interesting link between concavity and local extrema. Suppose a function f has a critical point c for which $f'(c) = 0$. Observe (as illustrated below) that f has a local minimum at c if its graph is concave up there. And f has a local maximum at c if it is concave down at c.



Local minimum at c

$f(x)$ is concave up

$f''(x) > 0$



Local maximum at c

$f(x)$ concave down

$f''(x) < 0$

Therefore there will be a local minimum at c if $f''(c)$ is positive, and a local minimum at c if $f'(c)$ is negative. This simple observation is called the **second derivative test** for identifying local extrema. It does the same thing as the first derivative test, but it uses the second derivative instead of the first derivative.

The Second Derivative Test (for finding local extrema of a function)

Suppose c is a critical point of $f(x)$ for which $f'(c) = 0$

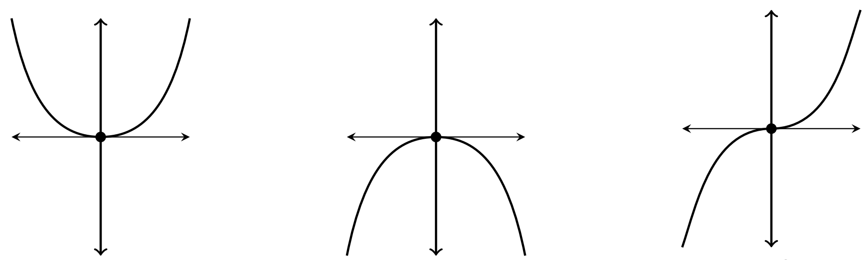
- If $f''(c) > 0$, then $f(x)$ has a local minimum at c.
- If $f''(c) < 0$, then $f(x)$ has a local maximum at c.
- If $f''(c) = 0$, then there is no conclusion. (Use first derivative test.)

The second derivative test can be (as we will see) easier to use than the first derivative test, but it does have some drawbacks. First, it only applies to critical points for which $f'(c) = 0$. If f has any critical points c for which $f'(c)$ is not defined, then the second derivative test says nothing about them.

Another drawback of the second derivative test is that it is inconclusive if $f''(c) = 0$. In this case there could be a minimum, a maximum, or no extremum at all at c, and the second derivative test can't distinguish between these possibilities. Figure 32.1 explains why this is so.

If ever the second derivative test is inconclusive (or if you have a critical point c for which $f'(c)$ is not defined), then you have to resort to the first derivative test.

Local min. at 0 | **Local max. at 0** | **No extremum at 0**



$f(x) = x^4$
$f'(x) = 4x^3$
$f''(x) = 12x^2$
$f''(0) = 0$

$f(x) = -x^4$
$f'(x) = -4x^3$
$f''(x) = -12x^2$
$f''(0) = 0$

$f(x) = x^3$
$f'(x) = 3x^2$
$f''(x) = 6x$
$f''(0) = 0$

Figure 32.1. If $f''(c) = 0$, the function f could have a local maximum, a local minimum, or neither at c. Thus $f''(c) = 0$ tells us nothing about extrema at c. Therefore the second derivative test is inconclusive when $f''(c) = 0$.

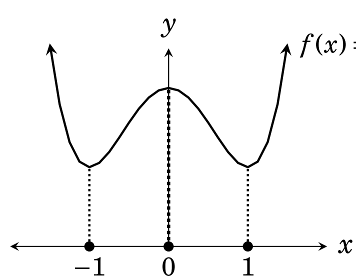
Let's work some examples using the second derivative test to find local extrema. We'll do the same examples we did in Chapter 31, but this time we'll use the second derivative test instead of the first derivative test.

Example 32.2 Find all local extrema of the function $f(x) = x^4 - 2x^2 + 2$.

Solution The derivative is $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$, from which we read off the critical points as -1, 0 and 1. Moreover, these critical points make f' zero, so the second derivative test will apply to them.

The second derivative is $f''(x) = 12x^2 - 4$. We now test each critical point.

Because $f''(-1) = 12(-1)^2 - 4 = 8 > 0$, there is a local minimum at -1.
Because $f''(0) = 12 \cdot 0^2 - 4 = -4 < 0$, there is a local maximum at 0.
Because $f''(1) = 12 \cdot 1^2 - 4 = 8 > 0$, there is a local minimum at 1.



This is exactly the same answer we got in Example 31.1. Notice how much easier our work was with the second derivative test. ✍

Example 32.3 Find all local extrema of $f(x) = \sqrt[3]{x}^2 - \frac{2}{3}x$ on $(-\infty, \infty)$.

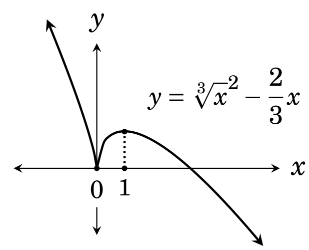
Solution We solved this using the first derivative test in Example 31.2, but now we will try it with the second derivative test. The derivative is

$$f'(x) = \frac{2}{3}x^{2/3-1} - \frac{2}{3} = \frac{2}{3}\left(x^{-1/3} - 1\right) = \frac{2}{3}\left(\frac{1}{\sqrt[3]{x}} - 1\right).$$

We can read off the critical points as 0 (because $f'(0)$ is undefined) and 1 (because $f'(1) = 0$). However, the second derivative test will not apply to the critical point 0. Nonetheless, let's apply it to the critical point 1.

The second derivative is $f''(x) = -\frac{2}{9}x^{-1/3-1} = -\frac{2}{9\sqrt[3]{x}^4}$. Because $f''(1) = -\frac{2}{9} < 0$, the second derivative test guarantees a local maximum at 1.

So the second derivative test has given us a partial answer. There is a local maximum at 1, but it didn't pick up the local minimum at 0. For that we need to use the first derivative test, as in Example 31.2. ✍



Example 32.4 Find all local extrema of $y = x\sin(x) + \cos(x) - \frac{x^2}{4}$ on $(-\pi, \pi)$.

Solution The domain of this function is all real numbers, but we are only asked about its extrema of it on $(-\pi, \pi)$. The derivative is

$$\frac{dy}{dx} = 1 \cdot \sin(x) + x\cos(x) - \sin(x) - \frac{x}{2} = x\left(\cos(x) - \frac{1}{2}\right).$$

As was noted in Example 31.3, the critical points are the values in $(-\pi, \pi)$ that make this zero, namely $x = 0$ and $x = \pm\frac{\pi}{3}$. The second derivative is

$$\frac{d^2y}{dx^2} = 1 \cdot \left(\cos(x) - \frac{1}{2}\right) - x\sin(x) = \cos(x) - \frac{1}{2} - x\sin(x).$$

Next, plug the critical points into the second derivative and note the signs.

$$\left.\frac{d^2y}{dx^2}\right|_{-\pi/3} = \cos\left(-\frac{\pi}{3}\right) - \frac{1}{2} - \left(-\frac{\pi}{3}\sin\left(-\frac{\pi}{3}\right)\right) = \frac{1}{2} - \frac{1}{2} - \frac{\pi}{3}\frac{\sqrt{3}}{2} < 0 \qquad \text{(negative)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{0} = \cos(0) - \frac{1}{2} - 0\sin(0) = \frac{1}{2} > 0 \qquad \text{(positive)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\pi/3} = \cos\left(\frac{\pi}{3}\right) - \frac{1}{2} - \frac{\pi}{3}\sin\left(\frac{\pi}{3}\right) = \frac{1}{2} - \frac{1}{2} - \frac{\pi}{3}\frac{\sqrt{3}}{2} < 0 \qquad \text{(negative)}$$

The function has local maxima at $-\frac{\pi}{3}$ and $\frac{\pi}{3}$, and a local minimum at 0. ✍

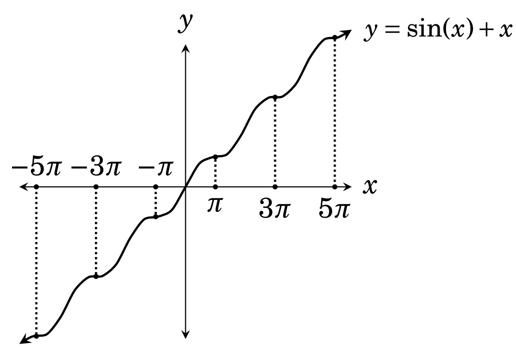
Example 32.5 Find all local extrema of $f(x) = \sin(x) + x$ on $(-\infty, \infty)$.

Solution We already solved this problem in Example 31.5, where we found that the function has no local extrema at all. Here we will attempt to do it again with the second derivative test.

The derivative $f'(x) = \cos(x) + 1$ is defined for all real values of x, so there are no critical points c for which $f'(c)$ is not defined. Therefore the critical points of f will be numbers for which the derivative $\cos(x) + 1$ equals 0. These are are the odd multiples of π (the numbers c for which $\cos(c) = -1$),

$$\ldots -7\pi,\ -5\pi,\ -3\pi,\ -\pi,\ \pi,\ 3\pi,\ 5\pi,\ 7\pi, \ldots$$

The second derivative is $f''(x) = -\sin(x)$, and because $f''(k\pi) = -\sin(k\pi) = 0$ for integer multiples of π, we conclude $f''(c) = 0$ for all critical points of f. Thus the second derivative test is entirely inconclusive. To solve this problem we must revert to the first derivative test, which was done in Example 31.5.



There we found that f continues rising forever, and has no extrema. ✍

Exercises for Chapter 32

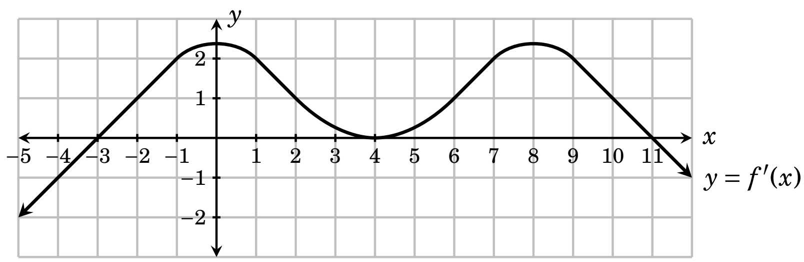
1. This problem concerns the function $f(x) = (x-2)e^x$. Find the intervals on which $f(x)$ increases/decreases. Find the intervals on which $f(x)$ is concave up/down.
2. This problem concerns the function $f(x) = 3x^{2/3} - 2x$. Find the intervals on which $f(x)$ increases/decreases. Find the intervals on which $f(x)$ is concave up/down.
3. Use the second derivative test to find the local extrema of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$.
4. Use the second derivative test to find the local extrema of $f(x) = \frac{3}{2}x^4 - x^6$.
5. Use the second derivative test to find the local extrema of $f(x) = 5x^4 + 20x^3 + 10$.
6. Use the second derivative test to find thelocal extrema of $f(x) = x^2 e^{-x}$.
7. Use the second derivative test to find the local extrema of $f(x) = x^2 e^x$.

8. Use the second derivative test to find thelocal extrema of $f(x) = x\ln|x|$.

9. Use the second derivative test to find the local extrema of $f(x) = e^{x^2-2x}$.

10. Use the second derivative test to find the local extrema of $y = \tan^{-1}\left(x^2+x-2\right)$.

11. The graph $y = f'(x)$ of <u>the derivative</u> of a function $f(x)$ is shown. Answer the questions <u>about $f(x)$.</u>



(a) State the intervals on which f increases and decreases.

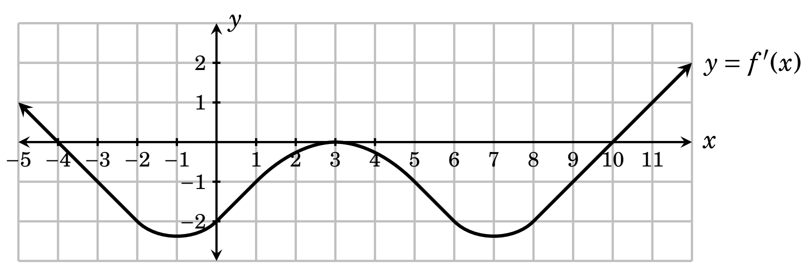
(b) List all critical points of f.

(c) At each critical point, say if f has a local maximum, minimum, or neither.

(d) State the intervals on which f is concave up and concave down.

(e) Based on this information, sketch a possible graph of f.

12. The graph $y = f'(x)$ of <u>the derivative</u> of a function $f(x)$ is shown. Answer the questions <u>about $f(x)$.</u>



(a) State the intervals on which f increases and decreases.

(b) List all critical points of f.

(c) At each critical point, say if f has a local maximum, minimum, or neither.

(d) State the intervals on which f is concave up and concave down.

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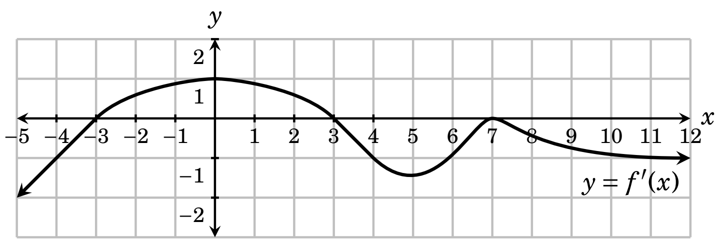
13. The graph $y = f'(x)$ of <u>the derivative</u> of a function $f(x)$ is shown. Answer the questions <u>about $f(x)$.</u>



(a) State the intervals on which f increases and decreases.

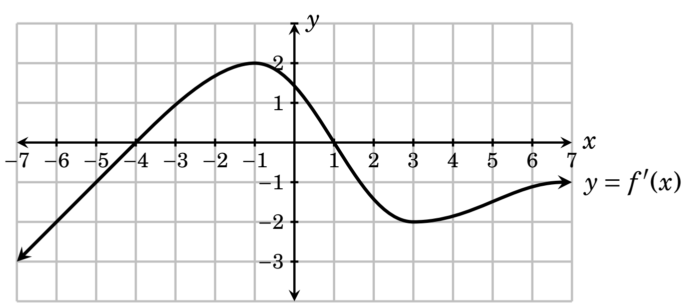
(b) List all critical points of f.

(c) At each critical point, say if f has a local maximum, minimum, or neither.

(d) State the intervals on which f is concave up and concave down.

(e) Based on this information, sketch a possible graph of f.

14. The graph $y = f'(x)$ of the derivative of a function $f(x)$ is shown. Answer the questions about $f(x)$.



(a) State the intervals on which f increases and decreases.

(b) List all critical points of f.

(c) At each critical point, say if f has a local maximum, minimum, or neither.

(d) State the intervals on which f is concave up and concave down.

(e) Based on this information, sketch a possible graph of f.

Exercises Solutions or Chapter 32

1. This problem concerns the function $f(x) = (x-2)e^x$. Find the intervals on which $f(x)$ increases/decreases. Find the intervals on which $f(x)$ is concave up/down.

By the product rule, $f'(x) = e^x + (x-2)e^x = e^x(x-1)$. This is negative on $(-\infty, 1)$ and positive on $(1, \infty)$. Therefore f decreases on $(-\infty, 1)$ and increases on $(1, \infty)$. The second derivative is $f''(x) = e^x(x-1) + e^x = xe^x$. This is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. Thus f is concave down on $(-\infty, 1)$ and concave up on $(0, \infty)$.

3. Use the second derivative test to find the local extrema of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$.

Since $f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x-1)(x+2)$, the critical points as $x = 1$ $x = 0$ and $x = -2$. The second derivative is $f''(x) = 36x^2 + 24x - 24$. Since $f''(1) = 36 \cdot 1^2 + 24 \cdot 1 - 24 = 36 > 0$, the function f has a a local minimum of $f(1) = -3$ at $x = 1$. Since $f''(0) = 36 \cdot 0^2 + 24 \cdot 0 - 24 = -24 < 0$, the function f has a a local maximum of $f(0) = 2$ at $x = 0$. Since $f''(-2) = 36 \cdot (-2)^2 + 24 \cdot (-2) - 24 = 72 > 0$, the function f has a a local minimum of $f(-2) = 34$ at $x = -2$.

5. Use the second derivative test to find the local extrema of $f(x) = 5x^4 + 20x^3 + 10$.

Since $f'(x) = 20x^3 + 60x^2 = 20x^2(x+3)$, the critical points as $x = 0$ and $x = -3$. The second derivative is $f''(x) = 60x^2 + 120x$. Since $f''(-3) = 60 \cdot (-3)^2 + 120 \cdot (-3) = 180 > 0$, the second derivative test says $f(x)$ has a local minimum of $f(-3) = -125$ at $x = -3$. However, $f''(0) = 60 \cdot 0^2 + 120 \cdot 0 = 0$, so the second derivative test gives no conclusion about a local extremum at 0. (Using the first derivative test, you will find that $f'(x) > 0$ on either side of 0, so there is no extremum at $x = 0$.)

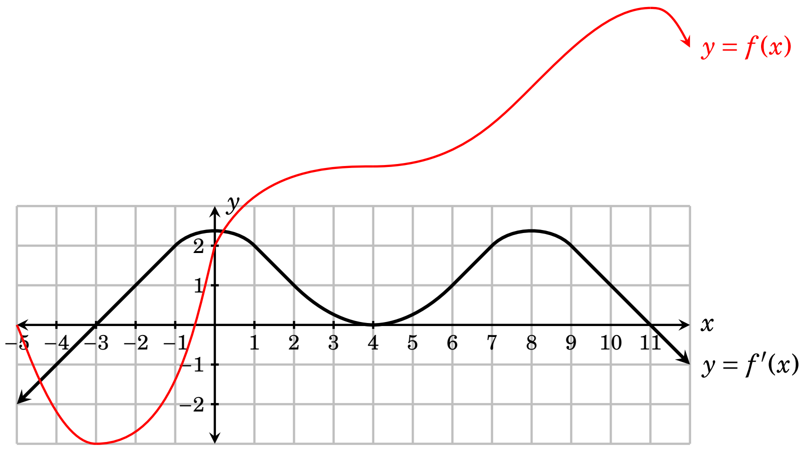
7. Use the second derivative test to find the local extrema of $f(x) = x^2e^x$.

Since $f'(x) = 2xe^x + x^2e^x = e^x(2x + x^2) = e^x x(2+x)$, we can read off the critical points as $x = 0$ and $x = -2$. The second derivative is $f''(x) = e^x(2x + x^2) + e^x(1 + 2x) = e^x\left(x^2 + 4x + 1\right)$. Since $f''(0) = e^0 = 1 > 0$, the function $f(x)$ has a local minimum of $f(0) = 0^2e^0 = 0$ at $x = 0$. Since $f''(-2) = e^{-2}\left((-2)^2 + 4(-2) + 1\right) = \frac{-3}{e^2} < 0$, the function $f(x)$ has a local maximum of $f(-2) = (-2)^2e^{-2} = \frac{4}{e^2}$ at $x = -2$.

9. Use the second derivative test to find the local extrema of $f(x) = e^{x^2-2x}$.

Since $f'(x) = e^{x^2-2x}(2x-2)$, the only critical point is $x = 1$. The second derivative is $f''(x) = e^{x^2-2x}(2x-2)(2x-2) + e^{x^2-2x}2 = e^{x^2-2x}\left(4x^2 - 8x + 6\right)$. Now observe that $f''(1) = e^{1^2-2\cdot 1}(4 \cdot 1^2 - 8 \cdot 1 + 6) = \frac{2}{e} > 0$. Thus the function f has a local minimum of $f(1) = e^{1^2-2} = \frac{1}{e}$ at $x = 1$.

11. The graph $y = f'(x)$ of the derivative of a function $f(x)$ is shown. Answer the questions about $f(x)$.



(a) $f(x)$ increases on $(-3,4)$ and $(4,11)$ because $f'(x) > 0$ there. Also, $f(x)$ decreases on $(-\infty,-3)$ and $(11,\infty)$ because $f'(x) < 0$ there.

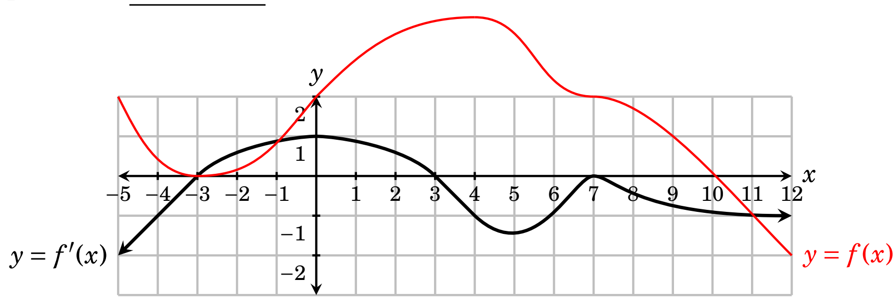
(b) The critical points of $f(x)$ are $-3,4,11$

(c) f has a local minimum at -3 because $f'(x)$ changes sign from $-$ to $+$ there. f has no extrema at $x = 4$ because $f'(x)$ does not change sign there.

(d) f is concave up on $(-\infty,0)$ and $(4,8)$ because f' is increasing on these intervals and consequently $f''(x) > 0$ there. Also, f is concave down on $(0,4)$ and $(8,\infty)$ because f' is decreasing on these intervals and consequently $f''(x) < 0$ there.

(e) A possible sketch of f is shown above.

13. The graph $y = f'(x)$ of the derivative of a function $f(x)$ is shown. Answer the questions about $f(x)$.



(a) The function f increases on $(-3,3)$ because $f'(x) > 0$ there. Also, f decreases on $(-\infty,-3)$, $(3,7)$ and $(7,\infty)$ because $f'(x)$ is negative on those intervals.

(b) The critical points of f are -3, 3 and 7.

(c) The function f has a local minimum at -3 because $f'(x)$ changes from $-$ to $+$ there. Also f has a local maximum at 3 because $f'(x)$ changes from $+$ to 1 there. There is no extremum at 7 because $f'(x)$ does not change sign there.

(d) The function f is concave up on $(-\infty,0)$ and $(5,7)$ because f' is increasing there (and hence $f''(x) > 0$). Also, f is concave down on $(0,5)$ and $(7,\infty)$ because f' is increasing there (and hence $f''(x) > 0$).

(e) A possible graph of f is sketched above.