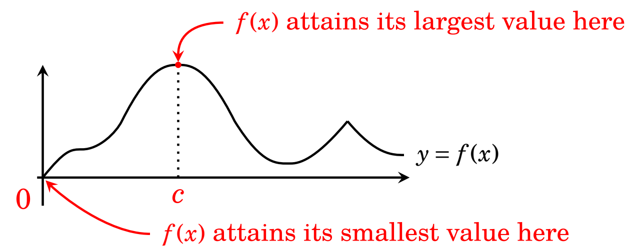
CHAPTER 33

Global Extrema

A local extremum of a function is not necessarily the highest or lowest point on its graph. A function may have a local maximum at c, and another, larger, local maximum somewhere else. We now consider *global extrema*, the absolute highest or lowest points on a function's graph.

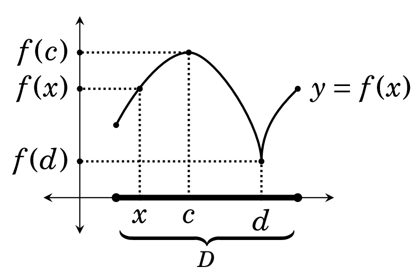


For example, above is a function that is defined on the interval $[0,\infty)$. It attains its largest value $f(c)$ at c. We say $f(c)$ is a *global maximum*. It attains its smallest value at 0, so we say $f(c)$ is a *global minimum*.

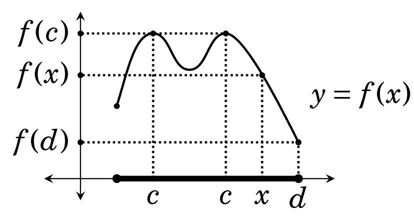
Definition 33.1 Suppose a function f is defined on a domain D. (Usually D is an interval.)

- $f(x)$ has an **global maximum** at $x=c$ if $f(c) \geq f(x)$ for all x in D.
- $f(x)$ has an **global minimum** at $x=d$ if $f(d) \leq f(x)$ for all x in D.

For example, the function to the right is defined on an interval D. It has a global maximum of $f(c)$ at c because $f(c) \geq f(x)$ for all x in D. It has a global minimum of $f(d)$ at d.

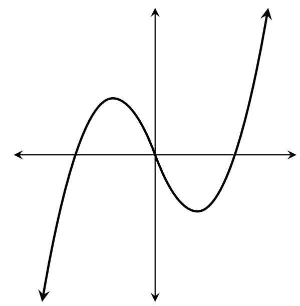


Here is a different function that has a global maximum of $f(c)$ that occurs at *two* different values of c. It has a global minimum of $f(d)$ at d (which happens to be an endpoint of the interval D).



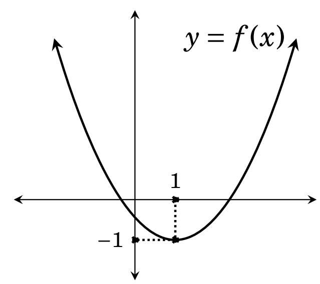
Global extrema are sometimes also called **absolute extrema**.

It is possible for a function to have no global (or absolute) extrema at all. The function on the right has no global maximum because for any point $(c, f(c))$ on the graph, there is always a higher point. Likewise, there is no global minimum because for any $(c, f(c))$ there is a lower point.

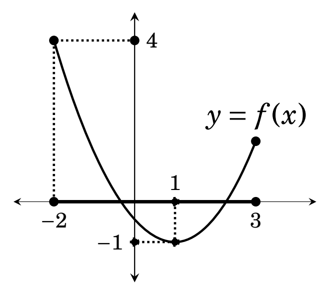


To further illustrate global extrema, we're now going to look at a few examples involving the same function defined on different intervals.

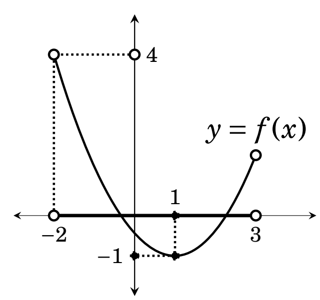
The quadratic function f on the right is defined on $D = (-\infty, \infty) = \mathbb{R}$. It has a global minimum of $f(1) = -1$ at $x = 1$. But this function has no global maximum $f(c)$ on D because no matter which c we pick, there is always some x in D for which $f(c) \leq f(x)$,



Now take the same function f, but restrict its domain to the **closed** interval $D = [-2, 3]$, as on the right. On D, f has a global minimum of $f(1) = -1$ at $x = 1$, and a global maximum of $f(-2) = 4$ at -2. (Notice that -2 happens to be an endpoint of D.)

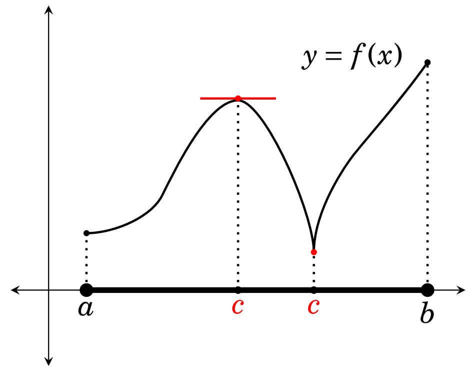


Next, here is the same function f, but defined on the **open** interval $D = (-2, 3)$. Again $f(1) = -1$ is a global minimum, at $x = 1$. But f has no global maximum $f(c)$ on D because if c is in D, then $f(c) < f(x)$ for any negative x between -2 and c. (It's tempting to say the maximum is $f(-2) = 4$, but $x = -2$ is not in D.)



The examples on this page have illustrated an important fact

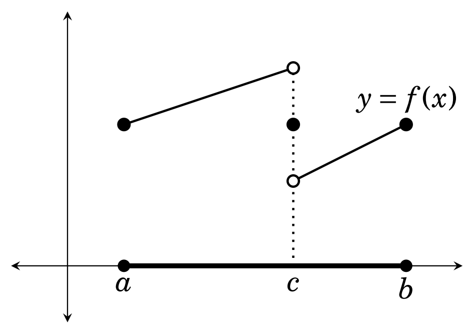
Fact 33.1 The absolute extrema of a function can depend on the interval on which it is defined.



But our examples may give you the sense that if f is defined on a *closed* interval $[a,b]$, then it must have both a global maximum and minimum. A global maximum (or minimum) could happen at a critical point c inside the interval, at a local maximum (or minimum). But if not, then we expect the highest (or lowest) point to be at one of the interval's endpoints a or b.

Actually, this turns out to be exactly the case, as the following theorem asserts. (We will not prove the theorem, but it should be intuitively clear.)

Theorem 33.1 If a function $f(x)$ is continuous on a closed interval $[a,b]$, then it has both a global maximum and a global minimum on $[a,b]$.



Though this theorem is certainly believable, its proof turns out to be surprisingly subtle. It relies not only on the structure of the interval (it is closed), but also on the fact that f is continuous. The graph on the right shows that the theorem falls apart if we remove continuity. This discontinuous f has no global extrema.

If we have a *differentiable* function f that is defined on a closed interval $[a,b]$, then it will also be continuous (by Theorem 18.1), and then the function will have both a global maximum and a global minimum (by Theorem 33.1). As noted above, we know where to look for these global extrema – at critical points and endpoints. This gives a procedure for finding the global extrema of a differentiable function on a closed interval.

How to find the global extrema of a function on a closed interval
Suppose a differentiable function f is defined on a closed interval $[a,b]$.

1. Find all critical points c of $f(x)$ in $[a,b]$.
2. Compute $f(a)$, $f(b)$, and $f(c)$ for each c in Step 1.
3. Largest value from Step 2 is absolute maximum.
   Smallest value from Step 2 is absolute minimum.

Example 33.1 Find the global extrema of $f(x) = x - 2\sqrt{x^2+9}$ on $[0,3]$.

Solution The first step is to find the critical points of f that are in $[0,3]$. The derivative is

$$f'(x) = 1 - \frac{2x}{\sqrt{x^2+9}}.$$

This is defined for all x, so all critical points are found by solving $f'(x) = 0$.

$$\begin{aligned} 1 - \frac{2x}{\sqrt{x^2+9}} &= 0 \\ 1 &= \frac{2x}{\sqrt{x^2+9}} \\ \sqrt{x^2+9} &= 2x \\ x^2+9 &= 4x^2 \\ 9 &= 3x^2 \\ 3 &= x^2 \end{aligned}$$

Thus the critical points are $\pm\sqrt{3}$, but only $\sqrt{3}$ is in $[0,3]$.

The second step is to evaluate f at this the interval endpoints and at $\sqrt{3}$.

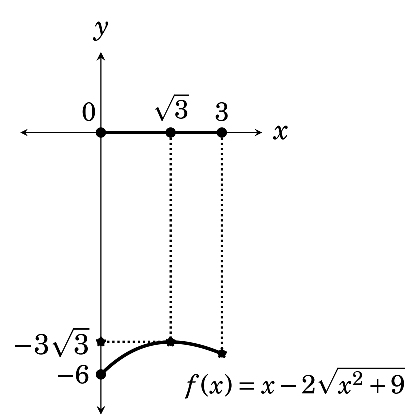
$$f(0) = 0 - 2\sqrt{0^2+9} = -6 \qquad \text{(global minimum)}$$

$$f(3) = 3 - 2\sqrt{3^2+9} = 3 - 2\sqrt{18} = 3 - 6\sqrt{2} \approx -5.48$$

$$f\left(\sqrt{3}\right) = \sqrt{3} - 2\sqrt{\sqrt{3}^2+9} = \sqrt{3} - 2\sqrt{12} = -3\sqrt{3} \approx -5.19 \qquad \text{(global maximum)}$$

Answer: On the interval $[0,3]$ the function $f(x) = x - 2\sqrt{x^2+9}$ has a global maximum of $-3\sqrt{3}$ at $x = \sqrt{3}$ and a global minimum of -6 at $x = 0$. ✍

Although it is not a necessary part of the solution of this exercise, it is instructive to look at a graph of $f(x) = x - 2\sqrt{x^2+9}$ on the interval $[0,3]$. Note that the global maximum happens at the critical point $x = \sqrt{3}$, and the global minimum is at the endpoint 0. The other endpoint 0 corresponds to neither a global maximum nor a global minimum.



Example 33.2 Find the global extrema of $f(x) = x^2\sqrt[3]{x-4}$ on $[0,5]$.

Solution The first step is to find the critical points of f that are in $[-1,5]$.

$$f'(x) = 2x\sqrt[3]{x-4} + x^2\frac{1}{3\sqrt{x-4}^2}.$$

Notice that $f'(4)$ is not defined, so $x = 4$ is a critical point of f in $[-1,5]$. The remaining critical points will be found by solving $f'(x) = 0$.

$$\begin{aligned} 2x\sqrt[3]{x-4} + x^2\frac{1}{3\sqrt{x-4}^2} &= 0 \\ \frac{6x(x-4)+x^2}{3\sqrt{x-4}^2} &= 0 && \text{(add fractions)} \\ \frac{x(7x-24)}{3\sqrt{x-4}^2} &= 0 \end{aligned}$$

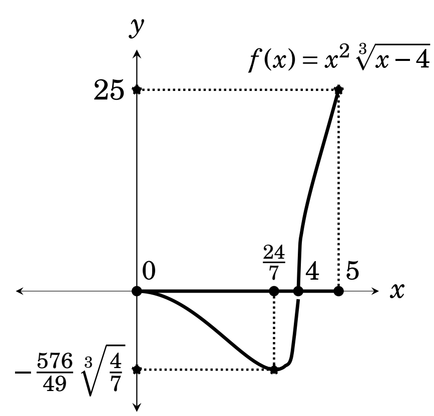
The derivative is zero for $x = 0$ and $x = \frac{24}{7}$. Thus 0, 4 and $\frac{24}{7}$ are the critical points in $[0,5]$. (Note that 0 is both a critical point and an endpoint.)

The second step is to evaluate $f(x)$ at the endpoints and critical points.

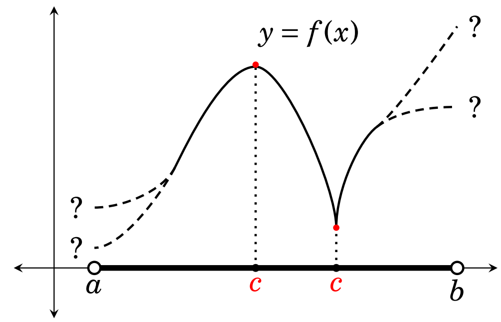
$$\begin{aligned} f(0) &= 0^2\sqrt[3]{0-4} = 0 \\ f(4) &= 4^2\sqrt[3]{4-4} = 0 \\ f\left(\frac{24}{7}\right) &= \left(\frac{24}{7}\right)^2\sqrt[3]{\frac{24}{7}-4} = -\frac{576}{49}\sqrt[3]{\frac{4}{7}} && \text{(global minimum)} \\ f(5) &= 5^2\sqrt[3]{5-4} = 25 && \text{(global maximum)} \end{aligned}$$

Answer: Selecting the smallest and largest of these values gives our answer: The function $f(x) = x^2\sqrt[3]{x-4}$ on $[-1,5]$ has a global minimum of $-\frac{576}{49}\sqrt[3]{\frac{4}{7}}$ at $x = \frac{24}{7}$ and a global maximum of 25 at $x = 5$.

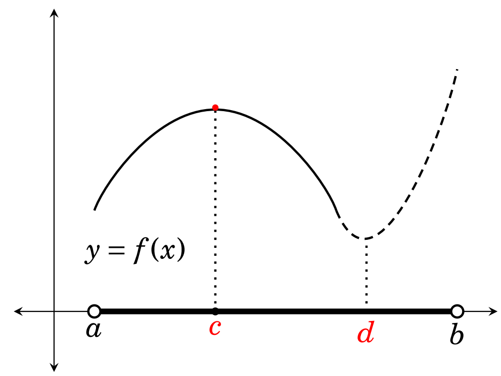
Although it is not a necessary part of the solution of this exercise, it is instructive to look at a graph of $f(x) = x^2\sqrt[3]{x-4}$. Observe that the critical points 0 and 4 correspond to horizontal and vertical tangents to the graph of f, respectively. The global extrema are found at the critical point $\frac{24}{7}$ and the endpoint 5.



So far in this chapter we have developed a failsafe procedure for finding the global extrema of a function on a closed interval. But we don't yet have a procedure that does this if the interval is *not closed*.



This can be tricky, as suggested by the function graphed on the right. This function f is defined on (a,b), but not at a and b. In such a circumstance, it may be unclear how f behaves near these endpoints. It may rise above any local maximum or dip below any local minimum – or not.



But in one case the answer is clearcut: Suppose f is defined on an open interval (a,b) and has **only one** local extremum on (a,b). For instance, say it has only a local maximum, at c, as shown on the right. Then the graph could never "turn around" and go higher than $f(c)$ because this would create a second local extremum, a minimum at some other point d in (a,b). Therefore $f(c)$ **must be a global maximum**.

The same reasoning applies if f has a local *minimum* at c:

Fact 33.2 Suppose f is continuous on an open interval, and f has only one local extremum on the interval, at c.

- If $f(c)$ is a local maximum, then it is a global maximum.
- If $f(c)$ is a local minimum, then it is a global minimum.

This gives a procedure for finding global extrema on an open interval in the case that there is only one local extremum.

How to find the global extrema of a function on an open interval
Suppose a function f is differentiable on an open interval (a,b).

1. Find the local extrema of f on (a,b).
2. Suppose Step 1 results in only one local extremum on (a,b).
   If it is a local maximum, then it is a global maximum.
   If it is a local minimum, then it is a global minimum.

(This only works if f has exactly one local extremum on (a,b).)

Despite this procedure's obvious limitations, we will find that it is the perfect tool for a great many applications.

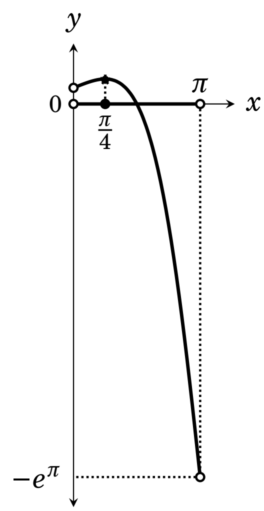
Example 33.3 Locate the global extrema of $f(x) = e^x \cos(x)$ on $(0, \pi)$.

Solution We first identify all local extrema. To find the critical points, we compute the derivative $f'(x) = e^x \cos(x) - e^x \sin(x) = e^x\big(\cos(x) - \sin(x)\big)$. This is defined for all x in $(0, \pi)$, and it is zero only at $x = \frac{\pi}{4}$. Therefore f has only one critical point $x = \frac{\pi}{4}$ on $(0, \pi)$.

Next, check to see if there is a local maximum or minimum at $\frac{\pi}{4}$. The second derivative is $f''(x) = e^x\big(\cos(x) - \sin(x)\big) + e^x\big(-\cos(x) - \sin(x)\big) = -2e^x \sin(x)$. As $f''(\pi/4) = -2e^{\pi/4} \sin(\pi/4) = 2e^{\pi/4}\frac{\sqrt{2}}{2} = -e^{\pi/2}\sqrt{2} < 0$, the second derivative test guarantees a local maximum for f at $\frac{\pi}{4}$.

Answer: We found only one local extremum for f on $(0, \pi)$, a local maximum. By Fact 33.2, f has a global maximum at $\frac{\pi}{4}$. It has no global minimum. ✍

It is not necessary to consider a graph of f in the above solution. (In fact, the calculus allows us to reach a conclusion about the highest point on the graph without drawing it.) However, it is still enlightening to look at the graph of $f(x) = e^x \cos(x)$ on $(0, \pi)$, which is shown on the right. Note the absolute maximum at $x = \frac{\pi}{4}$. Notice also that there would have been an absolute minimum of $-e^\pi$ at $x = \pi$ if the interval under consideration had been $[0, \pi]$ instead of $(0, \pi)$.



Exercises for Chapter 33

Find the global extrema of the stated function on the given interval.

1. $f(x) = x^2 - 4x + 7$ on $[0, 3]$

2. $f(x) = x^3 - 3x$ on $[0, 2]$

3. $f(x) = \cos(x)\sin(x)$ on $[0, \pi]$

4. $f(x) = \sqrt[3]{x}\big(x - 8\big)$ on $[-1, 27]$

5. $f(x) = \sqrt[3]{x}^4 + 4\sqrt[3]{x}$ on $[-8, 8]$

6. $f(x) = x^3 + x^2 - 5x + 2$ on $(-2, 0)$

7. $f(x) = \frac{3}{x} + x$ on $(0, 5)$

8. $f(x) = 2\sqrt{x} - x$ on $(0, 4)$

9. $y = \sin(x) - \dfrac{x}{2}$ on $(0, \pi/2)$

10. $y = x\sqrt{2-x}$ on $[-2, 2]$

11. $f(x) = \dfrac{16}{x} + x^2$ on $(0, \infty)$

12. $f(x) = xe^{-2x}$ on $(0, \infty)$

13. $f(x) = xe^{3x}$ on $(-5, \infty)$

14. $f(x) = \sin^2(x)$ on $[\pi, 2\pi]$

Exercises Solutions for Chapter 33

1. Find the global extrema of $f(x) = x^2 - 4x + 7$ on $[0, 3]$.

The derivative is $f'(x) = 2x - 4$. There is only one critical point, $x = 2$. Evaluating f at the endpoints and the critical point, we get: $f(0) = 7$, $f(2) = 3$ and $f(3) = 4$. The global maximum is 7 at $x = 0$ and the global minimum is 3 at $x = 2$.

3. Find the global extrema of $f(x) = \cos(x)\sin(x)$ on $[0, \pi]$.

The derivative is $f'(x) - \sin(x)\sin(x) + \cos(x)\cos(x) = \cos^2(x) - \sin^2(x)$. This is defined for all x, so all critical points will be solutions to $f'(x) = 0$. From $\cos^2(x) - \sin^2(x) = 0$ we get $\cos^2(x) = \sin^2(x)$, and then $\cos(x) = \pm\sin(x)$. The only values in $[0, \pi]$ for which this is true are $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. These are the critical points in $[0, \pi]$. Evaluating f at the endpoints and the critical point, we get: $f(0) = \cos(0)\sin(0) = 0$, $f(\pi) = \cos(\pi)\sin(\pi) = 0$, $f(\pi/4) = \cos(\pi/4)\sin(\pi/4) = \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = \frac{1}{2}$ and $f(3\pi/4) = \cos(3\pi/4)\sin(3\pi/4) = -\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = -\frac{1}{2}$. The global maximum is $\frac{1}{2}$ at $x = \frac{\pi}{4}$. The global minimum is $-\frac{1}{2}$ at $x = \frac{3\pi}{4}$.

5. Find the global extrema of $f(x) = \sqrt[3]{x}^4 + 4\sqrt[3]{x}$ on $[-8, 8]$.

The derivative is $f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}\left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x^2}}\right) = \frac{4}{3}\left(\frac{x+1}{\sqrt[3]{x^2}}\right)$. This equals 0 if $x = -1$ and it is undefined if $x = 0$. Therefore $x = -1$ and $x = 0$ are the only critical points, and they are in $[-8, 8]$. Next we evaluate f at the endpoints and the critical point: $f(-8) = \sqrt[3]{-8}^4 + 4\sqrt[3]{-8} = 16 - 8 = 8$, $f(-1) = \sqrt[3]{-1}^4 + 4\sqrt[3]{-1} = 1 - 4 = -3$, $f(0) = \sqrt[3]{0}^4 + 4\sqrt[3]{0} = 0$ and $f(8) = \sqrt[3]{8}^4 + 4\sqrt[3]{8} = 16 + 8 = 24$. Thus there is a global maximum of 24 at $x = 8$ and a global minimum of -3 at $x = -1$.

7. Find the global extrema of $f(x) = \frac{3}{x} + x$ on $(0, 5)$.

The derivative is $f'(x) = -\frac{3}{x^2} + 1$, which equals 0 at $x = \pm\sqrt{3}$. Thus there is only one critical point $x = \sqrt{3}$ in the open interval $(0, 5)$. Now, $f''(x) = \frac{6}{x^3}$, so $f''(\sqrt{3}) = \frac{6}{\sqrt{3}^3} > 0$. Therefore, by the second derivative test, f has a local minimum at $x = \sqrt{3}$. Because this is the only local extremum of f on $(0, 5)$, it is a global minimum. Thus f has a global minimum of $f(\sqrt{3}) = \frac{3}{\sqrt{3}} + \sqrt{3} = \frac{6}{\sqrt{3}}$ at $x = \sqrt{3}$. There is no global maximum.

9. Find the global extrema of $y = \sin(x) - \dfrac{x}{2}$ on $(0, \pi/2)$.

To find the critical points we solve $f'(x) = 0$, which is $\cos(x) - \dfrac{1}{2} = 0$. We get $\cos(x) = 1/2$, so the critical point is $x = \dfrac{\pi}{3}$. (This is the only value of x in $(0, \pi/2)$ for which $\cos(x) = 1/2$.)

As there is only one critical point in the interval, we just need to check if it gives a local max or min. For this we will use the second derivative test. The second derivative is $f''(x) = -\sin(x)$, and $f''\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} < 0$. Therefore f has a local maximum at $x = \frac{\pi}{3}$. **Answer:**

$$\boxed{f(x) = \sin(x) - \frac{x}{2} \text{ has a global max of } f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \text{ at } x = \frac{\pi}{3}. \text{ No global min.}}$$

11. Find the global extrema of $f(x) = \dfrac{16}{x} + x^2$ on $(0,\infty)$.

Solving $f'(x) = 0$, we get $-\dfrac{16}{x^2} + 2x = 0$. Multiply both sides by x^2 to get $-16+2x^3 = 0$, or $2x^3 = 16$, so $x^3 = 8$. Then the single critical point is $x = \sqrt[3]{8} = 2$. The second derivative is $f''(x) = \dfrac{32}{x^3} + 2$, so $f''(2) > 0$ and hence there is a global minimum at $x = 2$. There is no global maximum.

13. Find the global extrema of $f(x) = xe^{3x}$ on $(-5,\infty)$.

Solving $f'(x) = 0$, we get $e^{3x} + xe^{3x}3 = 0$, or $e^{3x}(1+3x) = 0$, hence the only critical point is $x = -1/3$. Now, $f''(x) = 3e^{3x}(1+3x) + e^{3x} \cdot 3 = 3e^{3x}(2+3x)$, and $f''(-1/3) = 3e^{3\cdot(-1/3)}(2+3\cdot(-1/3)) = 3e^{-1} > 0$. Thus there is a local minimum (hence global minimum) at $x = -1/3$, and the global minimum is of $f(-1/3) = -\frac{1}{3}e^{-1} = -\frac{1}{3e}$. There is no global maximum.