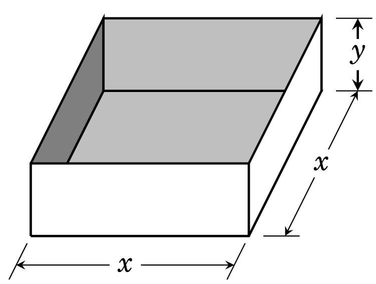
CHAPTER 34

Optimization Problems

In this chapter we will apply the ideas from Chapter 33 to solve real-world problems. The kinds of problems that we are concerned with here are called *optimization* problems. In an optimization problem the goal is to maximize some desirable quantity, or to minimize some undesirable quantity. For instance, if you are involved in some enterprise that incurs a monetary cost, then you want to execute a plan that *minimizes the cost*. In another instance you may want to maximize the effect of some medication. In certain situations this kind of problem can be solved with the methods that we have developed for finding global extrema.

Example 34.1 Imagine that you need to design a square concrete-lined pool having a box-like shape and a volume of 500 cubic feet.



In order to minimize costs you want the (concrete-lined) surface area to be as small as possible. What dimensions x and y result in a volume of 500 cubic feet, but with the smallest possible surface area?

Solution Our strategy will be to create a function $A(x)$ that gives the total surface area of the pool in terms of its length (and width) x. Then we will use what we've learned about finding global extrema to find the x that gives a global minimum for $A(x)$.

The area of the square bottom is x^2 square feet, and the area of each side is xy square feet. There are four sides, so the total surface area is

$$A = x^2 + 4xy \text{ square feet.}$$

Unfortunately this is not a function of x, because it contains the variable y. To overcome this, we will seek a way to express y in terms of x.

The volume of the box is length×width× height, which is x^2y cubic feet. And since the volume is required to be 500 cubic feet, this results in the equation

$$500 = x^2 y.$$

Such an equation is sometimes called a **constraint**. It allows us to solve for y in terms of x. Indeed, dividing both sides by x^2 yields $y = \frac{500}{x^2}$. Inserting this into the above formula $A = x^2 + 4xy$, we now have the pool's surface area expressed as the function

$$A(x) \;=\; x^2 + 4x\frac{500}{x^2} \;=\; x^2 + \frac{2000}{x}.$$

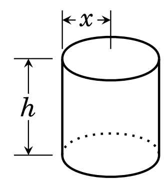
We want to find the global maximum of this function on the interval $(0,\infty)$ (since x is a length, it must be positive). To do this we first find the critical points. The derivative $A'(x) = 2x - \frac{2000}{x^2}$ is undefined at $x = 0$ but this is not a critical point because it's not in the domain of A. So to find all the critical points we must solve the equation $A'(x) = 0$.

$$\begin{aligned} 2x - \frac{2000}{x^2} &= 0 \\ 2x &= \frac{2000}{x^2} \\ 2x^3 &= 2000 \\ x^3 &= 1000 \\ x &= \sqrt[3]{1000} \;=\; 10 \end{aligned}$$

So there is only one critical point $x = 10$. Notice that $A''(x) = 2 + \frac{4000}{x^3}$ and so $A''(10) = 6 > 0$. By the second derivative test $A(x)$ has a local minimum at $x = 10$. Because this is the only local extremum on $(0,\infty)$, it is a global minimum. Therefore surface area of the pool is at a minimum when $x = 10$. For this value, our constraint gives $y = \frac{500}{10^2} = 5$.

Answer: To minimize surface area, the pool should be $10 \times 10 \times 5$ feet. ✍

Example 34.2 A cylindrical can with a bottom but no top is to be constructed using 100 square inches of metal. What should its radius and height be in order for the can to have the greatest possible volume?



Solution Denote the radius and height as x and h. In this problem we want to maximize volume. Our strategy is to find a function $V(x)$ giving the volume of the can in terms of the radius x. Then we can find which x yields a global maximum for volume $V(x)$. (Once x is found, we will then seek h.)

The volume of the cylinder is $V = \pi x^2 h$ (area of base, times height). But this contains both x *and* h. We want V to be a function of x alone, and not h.

To overcome this problem, let's look for a constraint. The one piece of information we have not used yet is that the surface area of the can is to be 100 square inches. The surface area of the circular bottom is πx^2 square inches. The surface area of the cylinder is $2\pi x h$ square inches (the circumference of the circular base times the height). Thus the total area is

$$100 = \pi x^2 + 2\pi x h.$$

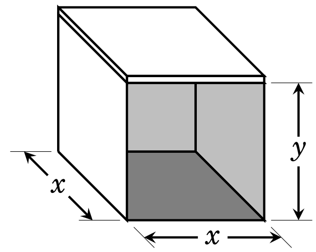
This constraint allows us to isolate h. Indeed, $h = \frac{100-\pi x^2}{2\pi x} = \frac{50}{\pi x} - \frac{x}{2}$. With this, our volume formula $V = \pi x^2 h$ becomes $V = \pi x^2\left(\frac{50}{\pi x} - \frac{x}{2}\right) = 50x - \frac{\pi}{2}x^3$. So the volume of the can is given by the function $V(x) = 50x - \frac{\pi}{2}x^3$. We seek the x that yields a global maximum of $V(x)$ on $(0,\infty)$. To do this we first find the critical points of V by examining the derivative $V'(x) = 50 - \frac{3}{2}\pi x^2$. Setting this equal to zero and solving,

$$\begin{aligned} 50 - \frac{3}{2}\pi x^2 &= 0 \\ 3\pi x^2 &= 100 \\ x &= \sqrt{\frac{100}{3\pi}} = \frac{10}{\sqrt{3\pi}}. \end{aligned}$$

As $V''(x) = -3\pi x < 0$ the second derivative test guarantees a local maximum of $V(x)$ at $x = \frac{10}{\sqrt{3\pi}}$. Since this is the only extremum, it is a global maximum.

Answer: Use dimensions $x = \frac{10}{\sqrt{3\pi}}$ and $h = \frac{50}{\pi 10/\sqrt{3\pi}} - \frac{10/\sqrt{3\pi}}{2}$. ✍

Example 34.3 You need to build a shed with an open front and square base (as illustrated), and containing a volume of 10,000 cubic feet.



The cost of materials are:
Roof: \$10 per square foot;
Floor: \$5 per square foot.
Walls: \$8 per square foot;
Find the dimensions x and y that will minimize the total cost of materials.

Solution In this problem the goal is to minimize cost of material. Our strategy is to create a function $C(x)$ that gives the total cost of materials if the base of the shed is x feet. Then we can use methods from the previous chapter to find an x that gives a global minimum for cost $C(x)$.

We begin building C by taking an inventory of the various costs.

Cost of roof: (x^2 square feet)×(\$10 per square foot) = ... \$$10x^2$
Cost of floor: (x^2 square feet)×(\$5 per square foot) = ... \$$5x^2$
Cost of walls: (3 walls)×(xy square feet)×(\$8 per square foot) = ... \$$24xy$

Total: \$ $15x^2+24xy$

So the total cost of materials is $15x^2+24xy$ dollars. Our strategy is to create a function $C(x)$ giving total cost of materials. But putting $C(x)=15x^2+24xy$ is not quite right because C is supposed to be a function of just one variable x. Here there are two variables x and y.

But remember that the volume of the shed is required to be 10,000 cubic feet. And since the volume of a box is length × width × height, we get $x^2y=10,000$. Hence $y=10,000/x^2$, Inserting this into $C(x)=15x^2+24xy$ gives

$$\begin{aligned} C(x) &= 15x^2+24x\frac{10,000}{x^2} \\ C(x) &= 15x^2+\frac{240,000}{x} \end{aligned}$$

Here x must be positive because it is length. So to solve the problem, we need to find the x that gives a global minimum of $C(x)$ on the interval $(0,\infty)$.

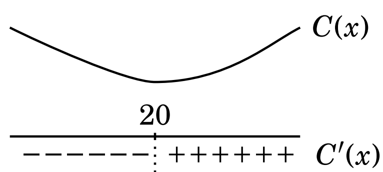
From here on our solution fits the familiar pattern of finding the global minimum of a function. The derivative is

$$C'(x) = 30x-\frac{240,000}{x^2}$$

To find the critical points we solve $C'(x) = 0$.

$$\begin{aligned}
30x - \frac{240,000}{x^2} &= 0 \\
30x &= \frac{240,000}{x^2} \\
30x^3 &= 240,000 \\
x^3 &= 8000 \\
x &= \sqrt[3]{8000} = 20
\end{aligned}$$

Therefore there is only one critical point, $x = 20$. This divides the domain $(0,\infty)$ of C into two intervals $(0,20)$ and $(20,\infty)$. Take the test point $x = 1$ in $(0,20)$ and note that $C'(1) = 30 - 24,000/1^2 < 0$, so $C'(x)$ is negative on $(0,20)$. And for a very large test point in $(20,\infty)$, like $x = 1000$, we have $C'(1000) = 30 \cdot 1000 - 24,000/1000^2 > 0$, so $C'(x)$ is positive on $(20,\infty)$.

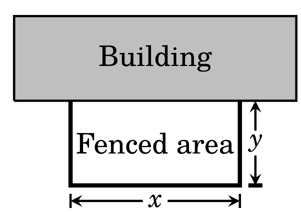


So the cost function $C(x)$ decreases before 20, and increases thereafter. Consequently we have a global minimum of cost if $x = 20$. Above we determined that $y = 10000/x^2$, so for $x = 20$ we get $y = 10000/20^2 = 10000/400 = 25$.

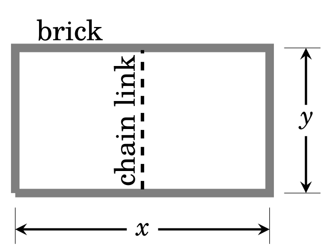
Answer: To minimize cost, use the dimensions $x = 20$ and $y = 25$. ✍

Exercises for Chapter 34

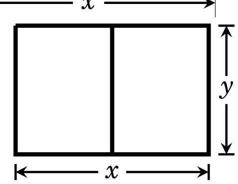
1. Suppose you have 160 feet of fencing material to enclose a rectangular region. One side of the rectangle will border a building, so no fencing is required for that side. Find the dimensions x and y that maximize the fenced area.



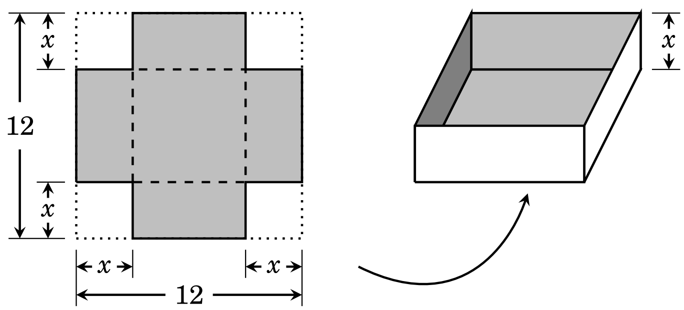
2. A total area of 2000 square feet is to be enclosed by two pens, as illustrated. The outside walls will be made of brick, and the inner dividing wall is chain link. The brick wall costs \$10 per foot, and the chain link costs \$5 per foot. Find the dimensions x and y that minimize the cost of construction.



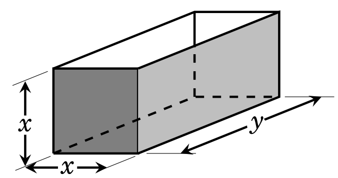
3. Suppose you have 120 feet of fencing material to enclose two rectangular pens, as shown. Find the dimensions x and y that maximize the total enclosed area.



4. An open-top box is made from a 12 by 12 inch piece of cardboard by cutting a square from each corner, and folding up. What should x be to maximize the volume of the box?

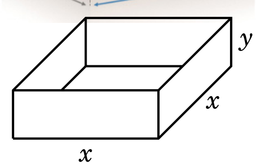


5. A metal box with two square ends and an open top is to contain a volume of 36 cubic inches. What dimensions x and y will minimize the total area of the metal surface?

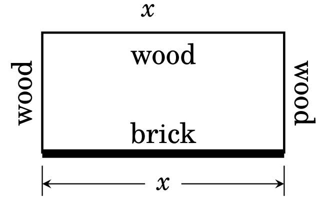


6. USPS rules say the length plus girth of a package cannot exceed 108 inches. (Girth = 2·width+2·height, as illustrated.) You must mail a package whose width and height are equal, and with the greatest possible volume. Find the dimensions of the package.

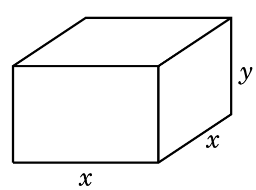
7. A cardboard box with a square base and open top is to have a volume of 4 cubic meters. Find the dimensions that result in a box that uses the least cardboard.



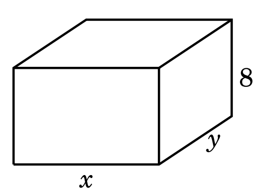
8. A rectangular region of 600 square meters needs to be enclosed by a fence. The south side of the region will be bounded by a brick wall, and the fencing on the remaining three sides will be made of wood. The brick wall is \$10 per meter, and the wood wall costs \$5 per meter. Find the length x of the brick wall that results in the lowest cost of materials.



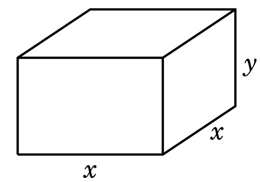
9. A tank with a square base is to be constructed to hold 10,000 cubic feet of water. The metal top costs \$6 per square foot, and the concrete sides and bottom cost \$4 per square foot. What dimensions x and y yield the lowest cost of materials?



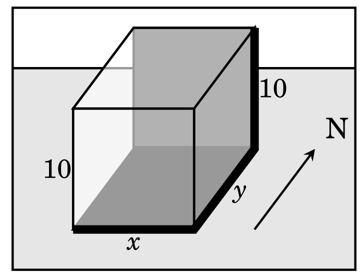
10. United Airlines restricts the size of carry-on packages. The policy states that the sum of the length, width, and height of a package cannot exceed 48 inches. What is the largest volume that an 8-inch high carry-on box can have?



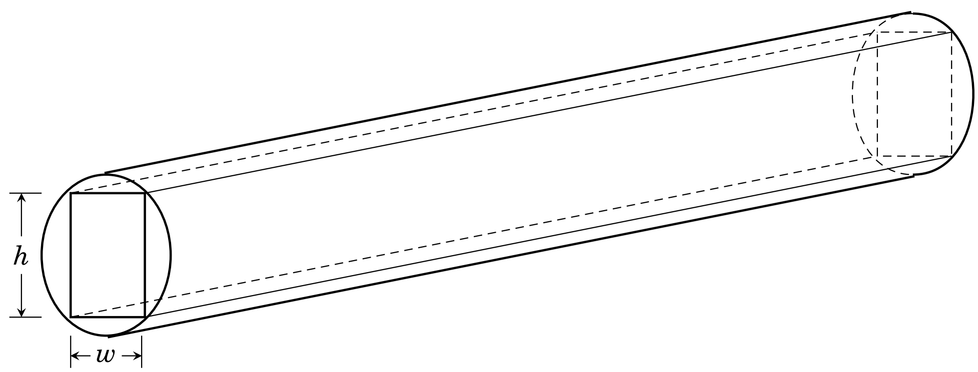
11. A box-shaped storage bin with a square base is to be constructed. The material for the top costs \$3 per square foot, the material for the bottom costs \$5 per square foot, and the material for the sides costs \$2 per square foot. Find the dimensions of the bin of maximum volume that can be constructed for \$2400.



12. You are designing a 10-foot-high box-shaped greenhouse of dimensions $x \times y \times 10$ (feet). It is required to have a volume of 10,000 cubic feet. The floor is to be made of concrete, and the north-facing wall is to be made of brick. The other three walls and the roof are to be made of glass. The concrete costs \$5 per square foot, the brick costs \$8 per square foot, and the glass costs \$2 per square foot. Find the dimensions x and y that minimize the cost of materials.



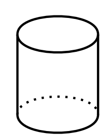
13. The strength of a rectangular beam is directly proportional to the product of its width and the square of its height. Find the dimension of the strongest beam that can be cut from a cylindrical log of diameter 10 inches.



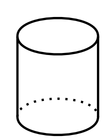
14. Find the point on the line $y = 2x + 3$ that is closest to the point $(5, 4)$.

15. Find two numbers x and y whose sum is 25 and for which $x^2 + 3y$ is minimized.

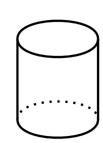
16. A cylindrical can with no top is to be constructed using 50 square inches of metal. What radius and height will produce a can with the greatest volume? (Note: although the can has no top, it does have a bottom.)



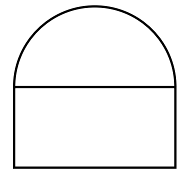
17. You are designing a cylindrical can which has a bottom but no lid. The can must have a volume of of 1000 cubic centimeters. What should the height and radius of the can be to minimize its surface area?



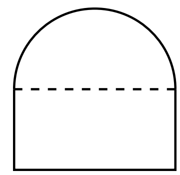
18. You are designing a cylindrical can (with both a top and a bottom) that must have a volume of of 1000 cubic centimeters. What should the height and radius of the can be to minimize its surface area?



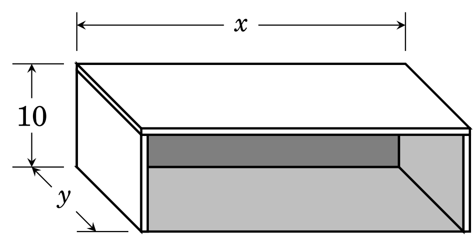
19. You are designing a window consisting of a rectangle with a half-circle on top, as illustrated. The client can only afford 1 meter of window framing material. The framing material runs around the outside of the window and between the rectangular and semicircular regions. What should the diameter of the half-circle be to maximize the area of the window?



20. You are designing a window consisting of a rectangle with a half-circle on top, as illustrated. The client can only afford 1 meter of window framing material which will run along the very outside portion of the window; no framing material is required between the rectangular and semicircular regions. What should the diameter of the half-circle be to maximize the area of the window?

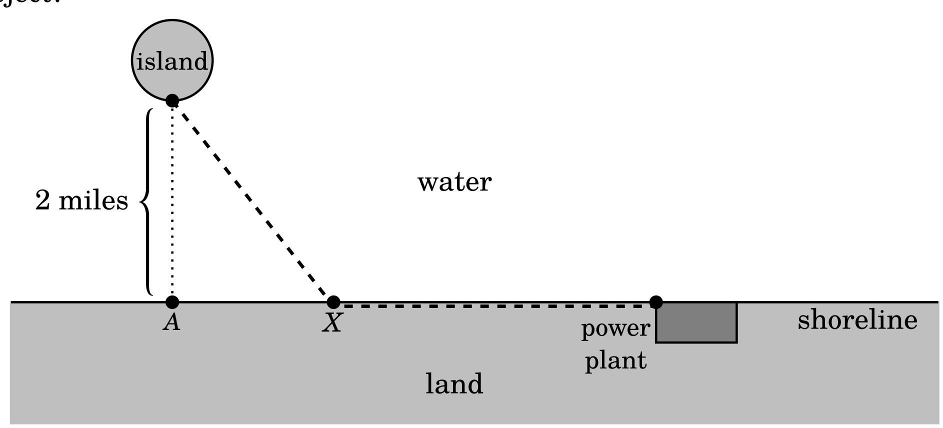


21. A 10-foot-high shed with flat roof and three walls is to be constructed, as illustrated. It is required that the shed have 800 square feet of floor space. Materials for the roof cost \$2 per square foot. Materials for the walls cost \$1 per square foot. The floor cost \$3 per square foot.



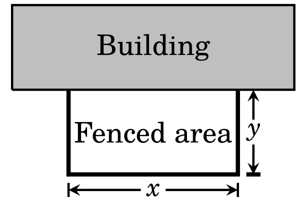
What dimensions x and y will minimize the total cost of materials?

22. An island is 2 miles from the nearest point A on a straight shoreline. Point A is 6 miles from a power plant. A utility company plans to lay electrical cable underwater from the island to the shore, and then along the shore to the power station. (As shown by the dashed line, below.) It costs \$2,000 per mile to lay the cable underwater, and \$1,000 per mile to run it along the shore. At what point X should the underwater cable meet the shore to minimize the cost of the project?



Exercise Solutions for Chapter 34

1. You have 160 feet of fencing material to enclose a rectangular region. One side of the rectangle will border a building, so no fencing is required for that side. Find the dimensions x and y that maximize the fenced area.

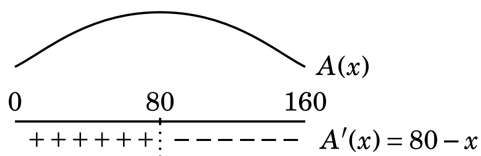


Solution In this problem the total length of the fence is $x+2y=160$. Thus we have the constraint $y=\frac{1}{2}(160-x)$.

We want to maximize area $A=xy$. Since $y=\frac{1}{2}(160-x)$, the enclosed area is a function $A(x)=xy=x\cdot\frac{1}{2}(160-x)=80x-\frac{1}{2}x^2$. Because x is the length, it must be positive and it cannot exceed the total available amount of 160 feet of fence. Thus in this problem $0<x<160$.

Therefore we want to find the x that gives a global maximum of $A(x)=80x-\frac{1}{2}x^2$ on the interval $(0,160)$.

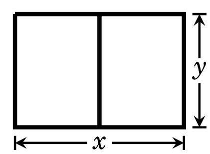
The derivative is $A'(x)=80-x$. To find the critical points, we solve $A'(x)=0$, that is, $80-x=0$, and this has one solution $x=80$. Thus the critical point is $x=80$.



Notice that $A'(x)$ is positive on $(0,80)$ and negative on $(80,160)$. Therefore $A(x)$ has a local maximum at 80, and since 80 is the only critical point, this must be a global maximum. Recall that $y=\frac{1}{2}(160-x)$, so when $x=80$, $y=40$.

Answer Greatest area is obtained by using $x=80$ and $y=40$.

3. Suppose you have 120 feet of fencing material to enclose two rectangular pens, as shown. Find the dimensions x and y that maximize the total enclosed area.

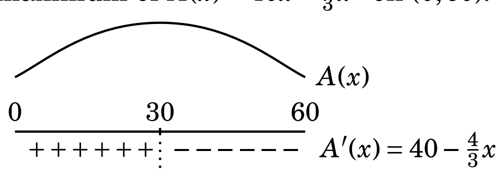


Solution The total length of fencing used is $120=2x+3y$, so $3y=120-2x$, and this yields the constraint $y=40-\frac{2}{3}x$.

We seek to maximize the enclosed area $A=xy=x\left(40-\frac{2}{3}x\right)=40x-\frac{2}{3}x^2$. Notice that x must be positive, and it cannot exceed $120/2=60$ feet.

Thus we seek the x that gives the global maximum of $A(x)=40x-\frac{2}{3}x^2$ on $(0,60)$.

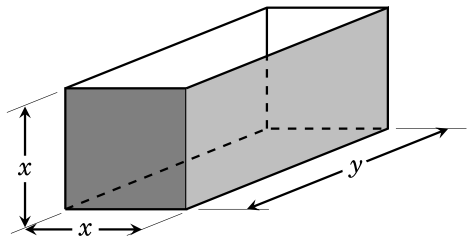
The derivative is $A'(x)=40-\frac{4}{3}x$. To find the critical points, we solve $A'(x)=0$, that is, $40-\frac{4}{3}x=0$, which has one solution $x=30$. Thus the critical point is $x=30$.



Notice that $A'(x)$ is positive on $(0,30)$ and negative on $(30,60)$. Therefore $A(x)$ has a local maximum at 30, and since 30 is the only critical point, this must be a global maximum. Recall that $y=40-\frac{2}{3}x$, so when $x=30$, $y=40-\frac{2}{3}\cdot 30=20$.

Answer: Greatest area is obtained by using $x=30$ and $y=20$.

5. A box with two square ends and an open top is to contain a volume of 36 cubic inches. What dimensions x and y will minimize the total area of the surface?



Solution The volume of the box must be $36 = x \cdot y \cdot x = x^2 y$, which means $y = \frac{36}{x^2}$.

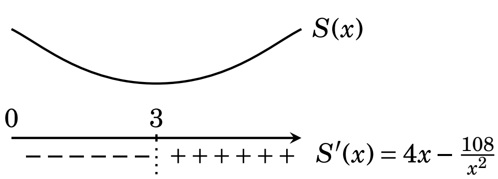
The front and back sides each contribute x^2 square inches to the surface area. The right and left sides each contribute $xy = x\frac{36}{x^2} = \frac{36}{x}$ square inches. The bottom also has an area of $xy = x\frac{36}{x^2} = \frac{36}{x}$ square inches. Therefore the total surface area of the box is $S(x) = 2x^2 + 3 \cdot \frac{36}{x} = 2x^2 + \frac{108}{x}$ square inches. Here x must be positive, but otherwise there are no restrictions on it. (As long as $y = \frac{36}{x^2}$, the volume will be 36 cubic inches.)

Therefore we seek the x that yields a global maximum of $S(x) = 2x^2 + \frac{108}{x}$ on $(0,\infty)$.

The derivative is $S'(x) = 4x - \frac{108}{x^2}$, which is defined for all x in $(0,\infty)$. So to find the critical points, we solve $S'(x) = 0$.

$$\begin{aligned} 4x - \frac{108}{x^2} &= 0 \\ 4x &= \frac{108}{x^2} \\ x^3 &= 27 \\ x &= \sqrt[3]{27} = 3 \end{aligned}$$

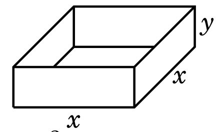
Thus there is only one critical point $x = 3$. This divides the domain into two intervals $(0,3)$ and $(3,\infty)$, Taking a test point of $x = 1$ in $(0,3)$, we get $S'(1) < 0$, so $S'(x)$ is negative on $(0,3)$. Taking a test point of $x = 10$ in $(3,\infty)$, we get $S'(10) > 0$, so $S'(x)$ is positive on $(3,\infty)$.



By the first derivative test, $S(x)$ has a local minimum at $x = 3$. Since 3 is the only critical point, this is a global minimum. As $y = \frac{36}{x^2}$, we get $y = \frac{36}{3^2} = 4$ when $x = 3$.

Answer: For smallest possible surface area, use the dimensions $x = 3$ and $y = 4$.

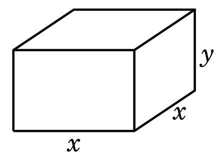
7. A cardboard box with a square base and open top is to have a volume of 4 cubic meters. Find the dimensions that result in a box that uses the least cardboard.



Solution We need to minimize the box's surface area, which is x^2+4xy square meters. Since the volume is to be 4 cubic meters, we get the equation $4 = xxy = x^2y$, so $\boxed{y=\frac{4}{x^2}}$. With this, the surface area becomes $x^2+4x\frac{4}{x^2} = x^2+\frac{16}{x}$. Hence we have to find the x that produces a global minimum of $S(x) = x^2+\frac{16}{x}$ on $(0,\infty)$. Since $S'(x) = 2x-\frac{16}{x^2}$, we can find the critical points by solving $2x-\frac{16}{x^2} = 0$. Multiplying both sides by x^2 gives $2x^3-16 = 0$, or $x^3 = 8$. Hence there is only one critical point on the interval, namely $x = 2$. Because $S''(x) = 2+\frac{32}{x^3}$, and $S''(2) = 6 > 0$, the function $S(x)$ has a local minimum at $x = 2$. Therefore surface area (cardboard) is minimized for $x = 2$ and $y = \frac{4}{2^2} = 1$.

Answer: For smallest possible surface area, use the dimensions $x = 2$ and $y = 1$.

9. A tank with a square base is to hold 10,000 cubic feet of water. The metal top costs \$6 per square foot, and the concrete sides and bottom cost \$4 per square foot. What dimensions x and y yield the lowest cost?



Solution The cost of the top is $6x^2$ dollars. The cost of the bottom is $4x^2$ dollars. The cost of each of the four sides is $4xy$ dollars. So the total cost of materials is

$$6x^2+4x^2+4\cdot 4xy \;=\; 10x^2+16xy \text{ dollars.}$$

This expression contains both an x and a y, so it is not a function of a single variable. To overcome this, notice that the tank's volume is $x\cdot x\cdot y = 10{,}000$ cubic feet. Therefore, $\boxed{y = 10{,}000/x^2}$, and the above cost of materials becomes

$$10x^2+16x\frac{10{,}000}{x^2} \;=\; 10x^2+\frac{160{,}000}{x} \text{ dollars.}$$

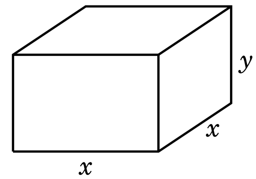
So we want to minimize the cost $C(x) = 10x^2+\dfrac{160{,}000}{x}$ on the interval $(0,\infty)$. First let's find the critical points.

$$\begin{aligned} 20x-\frac{160{,}000}{x^2} &= 0 && \text{(solve } C'(x)=0\text{)} \\ 20x^3-160{,}000 &= 0 && \text{(multiply both sides by } x^2\text{)} \\ x^3 &= 8000 \\ x &= \sqrt[3]{8000} = 20 \end{aligned}$$

There is only one critical point, $x = 20$. As $C''(x) = 20+320{,}000/x^3$, we get $C''(20) > 0$, and the second derivative test implies that $C(x)$ has a local minimum at $x = 20$. As 20 is the only critical point on the interval, we conclude that this is a a global minimum. For $x = 20$ we get $y = 10{,}000/20^2 = 25$ (see boxed equation above).

Answer: For lowest cost, use the dimensions $x = 20$ and $y = 25$.

11. A box-shaped storage bin with a square base is to be constructed. The material for the top costs \$3 per square foot, the material for the bottom costs \$5 per square foot, and the material for the sides costs \$2 per square foot. Find the dimensions of the bin of maximum volume that can be constructed for \$2400.



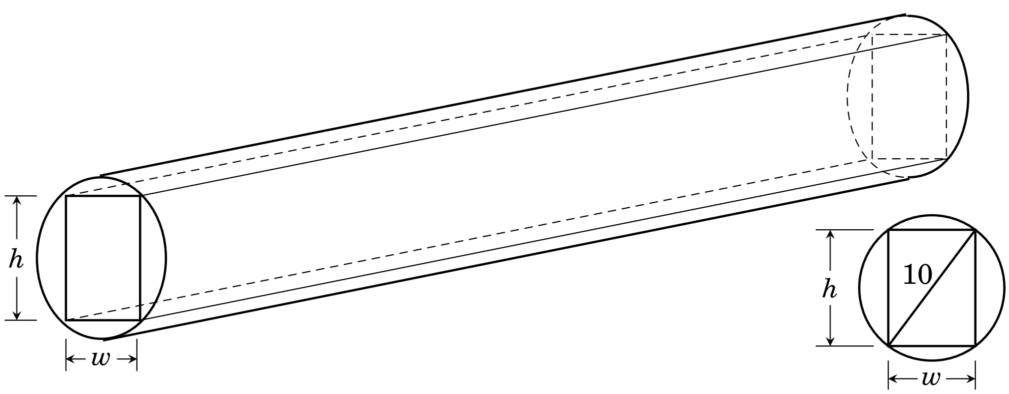
Solution Let the dimensions of the bin be x (length), x (width) and y (height). We need to find the dimensions that maximize volume $V = xxy = x^2y$. This is not a function of a single variable, so we need to express y in terms of x.

To do this, we seek a constraint. The cost of the top is $3x^2$ dollars, and the cost of the bottom is $5x^2$ dollars. The cost of each side is $2xy$ dollars. Therefore the total cost of materials is $3x^2+5x^2+4\cdot 2xy = 8x^2+8xy$ dollars. There is \$2400 to spend, which yields $8x^2+8xy = 2400$. Solving this for y gives $\boxed{y = \frac{2400-8x^2}{8x} = \frac{300-x^2}{x}.}$

Volume is now $V = x^2y = x^2\frac{300-x^2}{x} = 300x - x^3$. Thus we need to find the x that gives a global maximum of $V(x) = 300x - x^3$ on the interval $(0,\infty)$. Since $V'(x) = 300-3x^2$, we can see that there is only one critical point on the interval, namely $x = 10$. Now, $V''(x) = -6x$, and $V'(10) = -6\cdot 10 < 0$, so there is a local maximum at $x = 10$. As this is the only critical point, we infer that $V(x)$ has a global maximum at $x = 10$. For this x value, the boxed equation above yields $y = \frac{300-10^2}{10} = 20$.

Answer For maximum volume, use the dimensions $x = 10$ and $y = 20$,

13. The strength of a rectangular beam is directly proportional to the product of its width and the square of its height. Find the dimension of the strongest beam that can be cut from a cylindrical log of diameter 10 inches.



Solution A cross section is shown above. By the Pythagorean theorem, the height h and width w satisfy $w^2+h^2 = 10^2$, so $\boxed{h = \sqrt{100-w^2}}$.

The problem states that the strength of the beam is directly proportional to $wh^2 = w\sqrt{100-w^2}^2 = w(100-w^2) = 100w - w^3$. Thus to find the dimensions of the strongest beam we need to maximize $S(w) = 100w - w^3$ on the interval $(0,10)$. (The interval is $(0,10)$ because w must be positive but cannot exceed the diameter of the beam.) To find the critical point, we solve $S'(w) = 0$, which is $100-3w^2 = 0$. Then $w^2 = \frac{100}{3}$, so $w = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}}$ is the critical point. We will use the second derivative test to see if this yields a maximum. Since $S''(w) = -6w$, we get $S''\left(\frac{10}{\sqrt{3}}\right) = -6\frac{10}{\sqrt{3}} < 0$. Thus by the second derivative test $S(w)$ has a local maximum at $w = \frac{10}{\sqrt{3}}$, and since S has only one critical point on the interval, this must be a global maximum. Therefore, strength is maximized if $w = \frac{10}{\sqrt{3}}$ inches. By the boxed equation above, $h = \sqrt{100 - \left(\frac{10}{\sqrt{3}}\right)^2} = \sqrt{\frac{200}{3}} = 10\sqrt{\frac{2}{3}}$.

Answer Strength is maximized for $w = \frac{10}{\sqrt{3}}$ and $h = 10\sqrt{\frac{2}{3}}$.

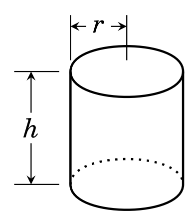
15. Find two numbers x and y whose sum is 25 and for which $x^2 + 3y$ is minimized.

Solution We need to minimize $x^2 + 3y$ subject to the constraint $x + y = 25$. Because $x+y = 25$ implies $y = 25-x$, the quantity we want to minimize is $x^2+3y = x^2+3(25-x) = x^2+75-3x$. Thus we need to find an x that gives a global minimum of $f(x) = x^2+75-3x$ on the interval $(-\infty,\infty)$ (because x could be any number). Since $f'(x) = 2x-3$, the only critical point is $x = 3/2$. Now, $f''(x) = 2$, so $f''(3/2) = 2 > 0$. By the second derivative test, f has a local minimum at $x = 3/2$. Since 3/2 is the only critical point, this is a global minimum.

Answer The quantity $x^2 + 3y$ is minimized when $x = 3/2$ and $y = 25 - 3/2 = 47/2$.

17. You are designing a cylindrical can which has a bottom but no lid. The can must have a volume of of 1000 cubic centimeters. What should the height and radius of the can be to minimize its surface area?

Solution Call the height of the can h and its radius r. The volume of a cylinder of height h and radius r is $V = \pi r^2 h$, so for this can we have $1000 = \pi r^2 h$. Consequently we get the constraint $h = \frac{1000}{\pi r^2}$.



The surface area of the circular bottom is πr^2. The surface area of the cylindrical side is the circumference of the bottom times the height h, that is, $2\pi rh$. Consequently the total surface area is $\pi r^2 + 2\pi rh$. Inserting the constraint $h = \frac{1000}{\pi r^2}$, the surface area is a function of r:

$$S(r) = \pi r^2 + 2\pi r\frac{1000}{\pi r^2} = \pi r^2 + \frac{2000}{r}.$$

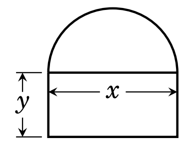
The radius r must be positive, meaning r must be in $(0,\infty)$. So we need to find the r (and h) that produce a global minimum of $S(r)$ on $(0,\infty)$. Note that $S'(r) = 2\pi r - \frac{2000}{r^2}$ is defined for all r in $(0,\infty)$. So to find all critical points we solve the equation $S'(r) = 0$:

$$\begin{aligned} 2\pi r - \frac{2000}{r^2} &= 0 \\ 2\pi r^3 - 2000 &= 0 \\ r^3 &= \frac{1000}{\pi} \\ r &= \frac{10}{\sqrt[3]{\pi}} \end{aligned}$$

Thus there is only one critical point $r = \frac{10}{\sqrt[3]{\pi}}$. The second derivative is $S''(r) = 2\pi + \frac{4000}{r^3}$, which is *positive* for all r in $(0,\infty)$. The second derivative test therefore guarantees a local minimum at the critical point $\frac{10}{\sqrt[3]{\pi}}$. Since this is the only critical point, there must be a global minimum there. Thus to minimize surface area we must use the dimensions $r = \frac{10}{\sqrt[3]{\pi}}$ and (by the constraint) $h = \frac{1000}{\pi\left(\frac{10}{\sqrt[3]{\pi}}\right)^2} = \frac{10}{\sqrt[3]{\pi}}$.

Answer: To minimize the can's surface area, both r and h should be $\frac{10}{\sqrt[3]{\pi}}$ cm.

19. You are designing a window consisting of a rectangle with a half-circle on top, as illustrated. The client (for some reason) can only afford 1 meter of window framing material. The framing material runs around the outside of the window and between the rectangular and semicircular regions. What should the diameter of the half-circle be to maximize the area of the window?



Solution Denote the diameter by x and the height of the rectangle by y. In this problem we want to maximize the total area. The area of the rectangle is xy (length × width), and the area of the semicircle is $\frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = \frac{\pi}{8}x^2$ (half the area of a circle of radius $\frac{x}{2}$). Thus our goal is to maximize total area $xy + \frac{\pi}{8}x^2$. This has two variables, and we need to express it as a function of *one* variable, so let's look for a constraint that expresses one variable in terms of the other.

The length of the arch is $\frac{1}{2}\pi x$ (half the circumference of a circle of diameter x). The perimeter of the rectangle is $2x+2y$. Thus the total length of the framing material is $\frac{1}{2}\pi x + 2x + 2y$. This has to be one meter so $1 = \frac{1}{2}\pi x + 2x + 2y$. Consequently $y = \frac{1}{2}\left(1 - \frac{1}{2}\pi x - 2x\right) = \frac{1}{2} - \left(1 + \frac{\pi}{4}\right)x$.

Thus the total area of the window is $xy + \frac{\pi}{8}x^2 = x\left(\frac{1}{2} - \left(1+\frac{\pi}{4}\right)x\right) + \frac{\pi}{8}x^2 = \frac{1}{2}x - \left(1+\frac{\pi}{8}\right)x^2$. This is a function $A(x) = \frac{1}{2}x - \left(1+\frac{\pi}{8}\right)x^2$, which we seek to maximize.

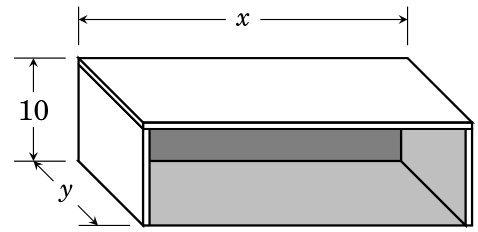
The derivative is $A'(x) = \frac{1}{2} - \left(2 + \frac{\pi}{4}\right)x$. To find the critical points we solve

$$\begin{aligned} \frac{1}{2} - \left(2 + \frac{\pi}{4}\right)x &= 0 \\ 2 - (8+\pi)x &= 0 \\ (8+\pi)x &= 2 \\ x &= \frac{2}{8+\pi} \end{aligned}$$

Thus A has only one critical point $x = \frac{2}{8+\pi}$. Since $A''(x) = \left(2 + \frac{\pi}{4}\right) < 0$ for all x, the second derivative test says A has a local maximum at $x = \frac{2}{8+\pi}$. Since there is only one critical point, this is a global maximum.

Answer: To maximize area, the diameter should be $= \frac{2}{8+\pi} \approx 0.179$ meters.

21. A 10-foot-high shed with flat roof and three walls is to be constructed, as illustrated. It is required that the shed have 800 square feet of floor space. Materials for the roof cost \$2 per square foot. Materials for the walls cost \$1 per square foot. The floor cost \$3 per square foot.



What dimensions x and y will minimize the total cost of materials?

Solution The roof costs $2xy$ dollars, and the floor costs $3xy$ dollars. The two side walls cost $1\cdot 10y$ dollars each, and the back wall costs $1\cdot 10x$ dollars. Thus the total cost of materials is $2xy+3xy+10y+10y+10x = 5xy+20y+10x$ dollars. This is not a function of a single variable, so we seek a constraint. The floor is required to have an area of 800 square feet, which means $xy = 800$, or $y = 800/x$. Substituting this into the above formula for cost gives the cost of materials as

$$C(x) = 5x\frac{800}{x} + 20\frac{800}{x} + 10x = 4000 + \frac{16000}{x} + 10x.$$

The domain of this function is $(0,\infty)$ because x must be positive, but otherwise can have any value so long as $y = 800/x$. Thus we need to find the x that produces a global minimum of $C(x)$ on $(0,\infty)$. To find any critical points, we solve $C'(x) = 0$.

$$\begin{aligned} -\frac{16000}{x^2} + 10 &= 0 \\ 10 &= \frac{16000}{x^2} \\ x^2 &= 1600 \\ x &= 40 \end{aligned}$$

As $C''(x) = 32000/x^3$, we get $C''(40) > 0$, the second derivative test says C has a local minimum at $x = 40$. But there is only one critical point, so this is a global minimum. The constraint $y = 800/x$ produces $y = 800/40 = 20$.

Answer: For least cost, use $x = 40$ and $y = 20$.