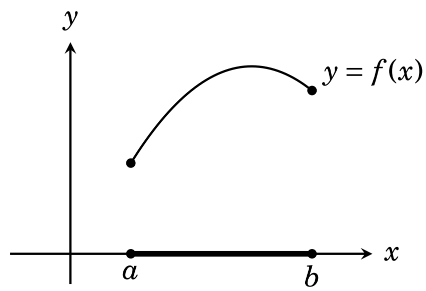
CHAPTER 35

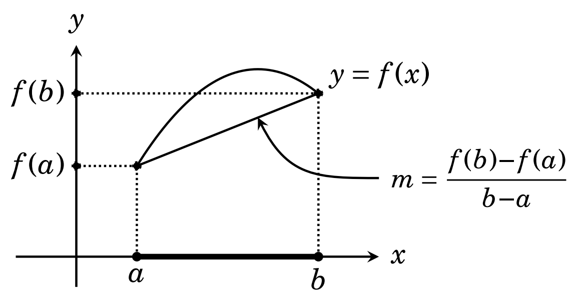
The Mean Value Theorem

This chapter's topic is called the *Mean Value Theorem*, or MVT. The MVT is not something (like, say, the chain rule) that you will use daily, but it does have some important consequences that we will address in Section 35.1.

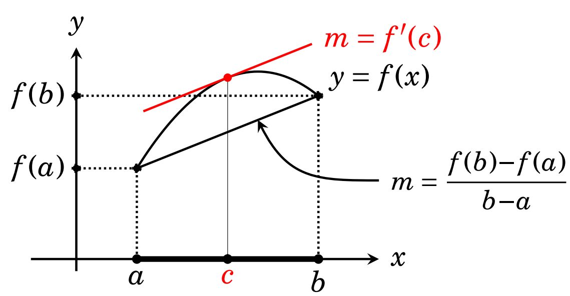
The MVT is easy to visualize. It concerns a function $f(x)$ that is defined on a closed interval $[a,b]$, as shown on the right. We're doing calculus, so say that $f(x)$ is differentiable, at least on (a,b), and continuous on $[a,b]$.



Draw a line segment between points $\big(a,f(a)\big)$ and $\big(b,f(b)\big)$. This line segment has slope $m = \frac{\text{rise}}{\text{run}} = \frac{f(b)-f(a)}{b-a}$.



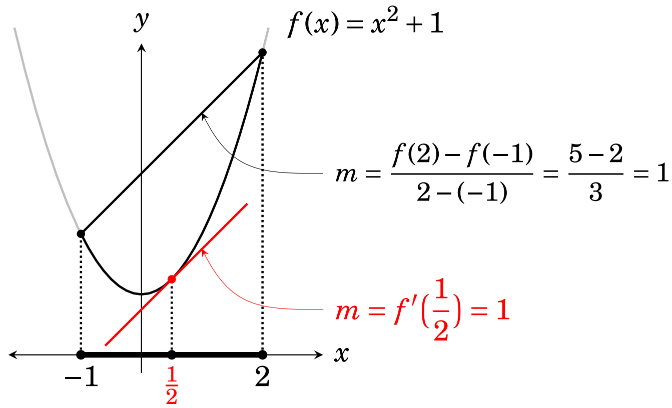
From this picture you would expect that at some point the tangent to $y = f(x)$ has the same slope as the line segment. In other words, there is a number c between a and b for which $f'(c) = \frac{f(b)-f(a)}{b-a}$. In fact, this is exactly what the mean value theorem says.



Fact 35.1 (Mean Value Theorem) If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is a number c in (a,b) for which

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

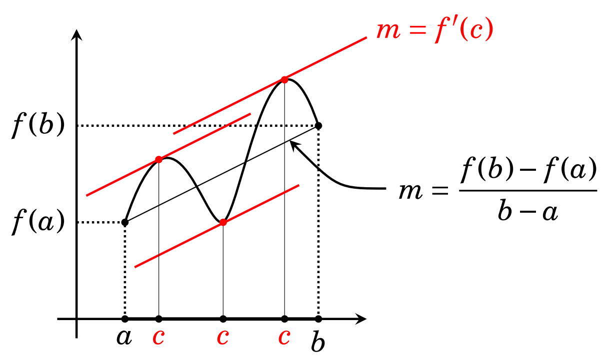
Example 35.1 Consider $f(x) = x^2 + 1$ on the interval $[a,b] = [-1,2]$, which is graphed below. Here $\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-1)}{2-(-1)} = \frac{5-2}{3} = 1$, so the mean value theorem states that there is a number c between -1 and 2 for which $f'(c) = 1$.



This number c for which $f'(c) = 1$ is easy to find. Because $f'(x) = 2x$, the equation $f'(c) = 1$ yields $2c = 1$, so $c = \frac{1}{2}$. The tangent line at $x = c = \frac{1}{2}$ is shown above, and indeed its slope is $f'\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{2} = 1$. ✍

Check your understanding by working Exercise 1.

Again, the mean value theorem asserts that if f is continuous on $[a,b]$ and differentiable on (a,b) then there is a number c in (a,b) for which $f'(c) = \frac{f(b)-f(a)}{b-a}$. It is possible that there is more than one such number c. For the function graphed below there happen to be *three* such numbers c. The mean value theorem guarantees that there is always at least one.



The mean value theorem can be proved easily with familiar techniques of finding global extrema. We will do this at the end of the chapter. But first we'll do one more example and then discuss some important consequences of the MVT.

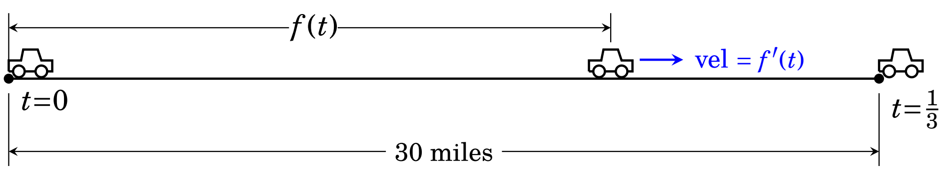
It so happens that in the situations in which the mean value theorem is most useful, there may not be enough information to actually find the number c (as we did in Example 35.1). The importance of the mean value theorem is that it guarantees that such a number c *exists*, even if we can't actually *find* it. Such is the case in the next example.

Example 35.2 The purpose of this example is to convince you that—even long before you studied calculus—you've had an intuitive, ingrained understanding of the mean value theorem.

In this thought experiment, imagine driving 30 miles in 20 minutes (⅓ hour) along a straight road. Say the speed limit is 70 mph. The question we will ask is this: During the trip, did you ever exceed the speed limit?

Intuition says **yes:** Your average velocity for the trip was $\frac{30 \text{ miles}}{\frac{1}{3} \text{ hour}} = 90$ mph. Thus at some point you were driving 90 mph or higher. You were speeding.

This idea—that your *average* (or *mean*) velocity equals your *exact* velocity at some instant—is really just an instance of the mean value theorem. To see how, let $f(t)$ be the position function, giving your distance from the starting point at time t. Under this setup, we know that your velocity at time t is $f'(t)$ mph.



Notice that $f(t)$ is defined for times $0 \le t \le 1/3$, that is, f is defined on the interval $[a,b] = [0,1/3]$. The mean value theorem says that for some time $t = c$ in $[0,1/3]$

$$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}} = \frac{f(\frac{1}{3})-f(0)}{\frac{1}{3}-0} = \frac{30-0}{\frac{1}{3}-0} = 90 \text{ mph.}$$

(exact velocity at time $t{=}c$)

(average [mean] velocity of trip)

Thus the mean value theorem simply asserts that the average rate of change of f on $[a,b]$ equals exact rate of change at some c. ✍

35.1 Consequences of the Mean Value Theorem

Although the mean value theorem itself will not play a big role in Calculus I, it *does* have two significant consequences, which we now discuss.

Both of these consequences are very believable, and should come as no surprise. However, the MVT nails them down beyond the shadow of a doubt. But if the consequences seem obvious to you, there's no harm in skipping their proofs. It is more important to understand and internalize the consequences themselves, rather than the fine points of their proofs.

The first consequence concerns functions $f(x)$ for which $f'(x) = 0$. We know that if $f(x)$ is a constant function, then its derivative is zero. But what about the other way around? If $f'(x) = 0$, then must $f(x)$ be a constant function? Our first MVT consequence says yes.

Fact 35.2 If $f(x)$ is differentiable on an interval (a,b), and $f'(x) = 0$ for all x in (a,b), then f is a constant function $f(x) = C$ on (a,b).

Proof. Suppose $f(x)$ meets the stated conditions. Fix a number d in (a,b). Let $C = f(d)$. We'll show below that $f(x) = C$ for any x in (a,b).

So take any x in (a,b). Assume first that x is to the *left* of the number d. Because f is differentiable on (a,b) it is also differentiable on the interval (x,d), and continuous on $[x,d]$ (because differentiability implies continuity). By the MVT, there is a number c in $[x,d]$ with

$$f'(c) = \frac{f(d)-f(x)}{d-x}.$$

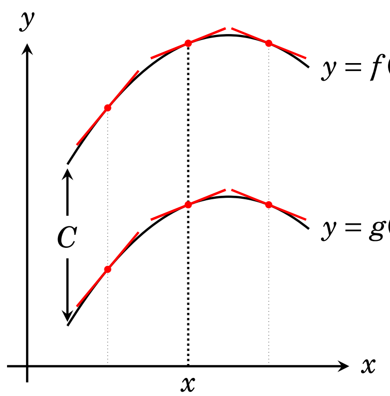
Because $f'(x) = 0$ on (a,b), we know $f'(c) = 0$, and the above becomes

$$\begin{aligned} 0 &= \frac{f(d)-f(x)}{d-x} \\ 0\cdot(d-x) &= f(d)-f(x) \\ 0 &= f(d)-f(x) \\ f(x) &= f(d) = C. \end{aligned}$$

We've just shown that $f(x) = C$ for any x in (a,b) that is to the left of d. If x is to the *right* of d, just repeat the above argument, replacing the interval (x,d) with (d,x). Again, we get $f(x) = C$. ■

In short, Fact 35.2 says: *If the derivative of a function is zero, then the function is a constant.* We sometimes downplay the interval, leaving it unspecified or regarding it simply as $(a,b) = (-\infty,\infty) = \mathbb{R}$ if appropriate.

Our second significant consequence of the mean value theorem involves two functions that have the same derivative. For example, consider the functions $f(x)$ and $g(x)$ graphed below. At any point x they have equal slopes, that is $f'(x) = g'(x)$.



Thus the rise and fall of $f(x)$ echos that of $g(x)$; the graph of $f(x)$ looks just like the graph of $g(x)$ except that it's slightly higher. You would guess that $f(x) = g(x)+C$ for some constant C. That is exactly what our second MVT consequence says.

Fact 35.3 Suppose two functions $f(x)$ and $g(x)$ are differentiable on an interval (a,b). If $f'(x) = g'(x)$ for all x in (a,b), then $f(x) = g(x)+C$ for some constant C.

Proof. Suppose $f'(x) = g'(x)$ for all x in (a,b). Let $h(x) = f(x)-g(x)$. Then $h'(x) = f'(x)-g'(x)$ on (a,b). Because $f'(x) = g'(x)$, this becomes $h'(x) = 0$ on (a,b). Then by Fact 35.2, $h(x) = C$ for some constant C. This means $f(x)-g(x) = C$, so $f(x) = g(x)+C$. ■

In short, Fact 35.3 says: *Two functions with the same derivative differ by a constant.* Again, we sometimes downplay the interval, regarding as unspecified or simply as $(a,b) = (-\infty,\infty) = \mathbb{R}$. But technically the interval has significance, because not all functions have domain $\mathbb{R}$.

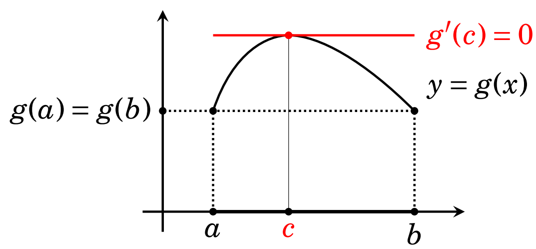
It is also important that f and g be defined on a *single* interval (a,b). If they are defined on *more than one interval*, then Fact 35.3 may break down. (See this chapter's Exercise 6.)

Likewise if f is not defined on a *single* interval (a,b), then Fact 35.2 may not hold. Exercise 5 asks for an example of function f with domain $(-\infty,0)\cup(0,\infty)$, such that f is non-constant but $f'(x) = 0$.

35.2 Proof of the Mean Value Theorem

So far in this chapter we've stated the mean value theorem, explained its meaning and stated two of its significant consequences. We have not yet actually *proved* the MVT, and we will do so now. However, if you feel that the MVT and its consequences are obvious (or if you're content to accept it as fact), then you can skip this section.

Our proof will involve a preliminary result known as *Rolle's theorem*. In essence, Rolle's theorem says that if a function $g(x)$ has the property that $g(a) = g(b)$ then there is a number c between a and b for which $g'(c) = 0$. (See the picture below. The graph of g starts at height $g(a)$, and moves higher. But it has to go back down to height $g(b)$, so it must "top out" at some c where $g'(c) = 0$.)



Fact 35.4 (Rolle's Theorem)
Suppose $g(x)$ is continuous on $[a,b]$, and differentiable on (a,b).
If $g(a) = g(b)$, then there is a number c in (a,b) for which $g'(c) = 0$.

Proof. Let g be as stated, and suppose $g(a) = g(b)$. By the extreme value theorem (Fact 33.1), g has both a global maximum and minimum on $[a,b]$.

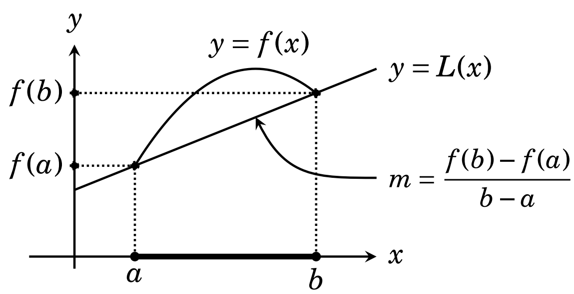
If a global maximum or minimum occurs at a point c in (a,b), then c must be a critical point of g. This means that either $g'(c)$ is undefined, or $g'(c) = 0$. But g is differentiable on (a,b), so $g'(c)$ cannot be undefined. Therefore $g'(c) = 0$.

On the other hand, if neither the global maximum nor minimum occurs inside (a,b), then the maximum occurs at one endpoint of $[a,b]$ and the minimum at the other. But since $g(a) = g(b)$, this means the absolute maximum and absolute minimum are equal, so g must be a constant function on $[a,b]$. Then $f'(c) = 0$ for *any* c in (a,b). ■

The mean value theorem follows quickly from Rolle's theorem. For convenience re restate it below.

> **Fact 35.1** (Mean Value Theorem)
> If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), then there is a number c in (a,b) for which $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof. Say $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). We now explain why the stated number c must exist.



Consider the straight line passing through the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$. (See the diagram above.) This line is the graph of a linear function $L(x)$. The derivative of $L(x)$ equals the slope of the line, that is,

$$L'(x) = \frac{f(b)-f(a)}{b-a} \qquad (*)$$

for any x. Now let $g(x)$ be the function

$$g(x) = f(x) - L(x).$$

Notice that g is differentiable on (a,b) because both f and L are, and g is continuous on $[a,b]$ because both f and L are. Also notice that $g(a) = 0 = g(b)$. Therefore, g meets the conditions of Rolle's theorem, which asserts that there is a number c in (a,b) for which

$$g'(c) = 0.$$

Because $g'(c) = f'(c) - L'(c)$, the above equation becomes

$$\begin{aligned} f'(c) - L'(c) &= 0 \\ f'(c) - \frac{f(b)-f(a)}{b-a} &= 0 && \text{(by } (*)\text{)} \\ f'(c) &= \frac{f(b)-f(a)}{b-a}. \end{aligned}$$

This proves the mean value theorem. ■

Exercises for Chapter 35

1. Consider the function $f(x) = x^3 - 3x$ on the interval $[0,3]$. Find the number c in $(0,3)$ guaranteed by the mean value theorem.

2. Consider the function $\ln(x)$ on the interval $[1,5]$. Find the number c in $(1,5)$ guaranteed by the mean value theorem.

3. Consider the function $f(x) = x^3 - x$ on the interval $[-2,3]$. Find *all* numbers c in $(-2,3)$ guaranteed by the mean value theorem.

4. The record for weight loss in a human is a drop from 487 pounds to 130 pounds over an eight month period. Use the mean value theorem to show that the rate of weight loss exceeded 44 pounds per month at some time during the eight months.

5. Find an example of a function $f(x)$, with domain $(-\infty,0)\cup(0,\infty)$, for which $f'(x) = 0$ but $f(x)$ is not a constant function. (This example shows that Fact 35.2 can fail if f is not defined on a *single* interval (a,b).)

6. Find two functions $f(x)$ and $g(x)$, each with domain $(-\infty,0)\cup(0,\infty)$, for which $f'(x) = g'(x)$, but $f(x) \neq g(x) + C$. (This example shows that Fact 35.3 can fail if f and g are not defined on a *single* interval (a,b).)

Solutions for Chapter 35

1. Consider $f(x) = x^3 - 3x$ on $[0,3]$. Find the number c guaranteed by the MVT.

Solution We seek a number c such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(0)}{3-0} = \frac{18-0}{3} = 6$. Since $f'(x) = 3x^2 - 3$, we seek a c for which $3c^2 - 3 = 6$, which reduces to $c^2 = 3$. Therefore $c = \pm\sqrt{3}$. Of these two values, only $c = \sqrt{3}$ is in $[0,3]$. Therefore $c = \sqrt{3}$ is the number in $(0,3)$ for which $f'(c) = \frac{f(3)-f(0)}{3-0}$.

3. Consider $f(x) = x^3 - x$ on $[-2,3]$. Find *all* numbers c in $(-2,3)$ guaranteed by the mean value theorem.

Solution We seek a number c such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(-2)}{3-(-2)} = \frac{24-(-6)}{5} = 6$. Since $f'(x) = 3x^2 - 1$, we seek a c for which $3c^2 - 1 = 6$, which reduces to $c^2 = \frac{7}{3}$. Therefore $c = \pm\sqrt{\frac{7}{3}}$. Both of these numbers are in $[-2,3]$. They are the numbers c in $(-2,3)$ for which $f'(c) = \frac{f(3)-f(-2)}{3-(-2)}$.

5. Find an example of a function $f(x)$, with domain $(-\infty,0)\cup(0,\infty)$, for which $f'(x) = 0$ but $f(x)$ is not a constant function.

Solution Let $f(x) = \begin{cases} 5 & \text{if } x < 0 \\ 3 & \text{if } x > 0. \end{cases}$

This has domain $(-\infty,0)\cup(0,\infty)$. It is not a constant function, but $f'(x) = 0$.