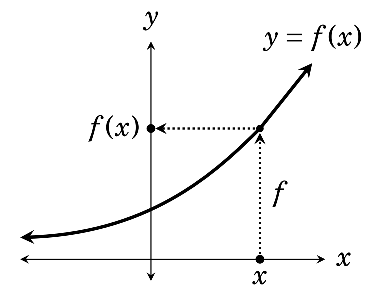
CHAPTER 4

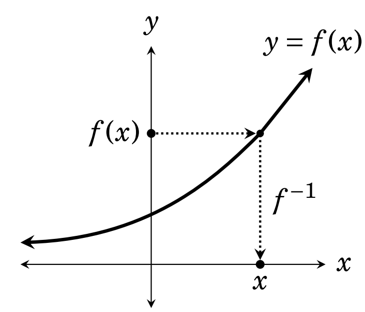
Inverse Functions

Under the right circumstances, a function f will have a so-called *inverse*, a function f^{-1} that "undoes" the effect of f. Whereas f sends an input x to the number $f(x)$, the function f^{-1} sends the number $f(x)$ back to x. We describe f^{-1} intuitively below before giving an exact definition.

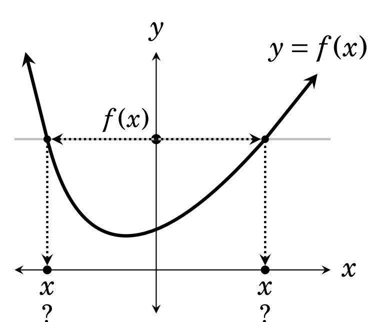
First consider a function f. From its graph, we can describe the rule f as follows. For an input x, to find the corresponding output value $f(x)$, move vertically from point x on the x-axis until reaching the graph of f. Then move horizontally to reach the output number $f(x)$ on the y-axis.



The inverse function f^{-1} reverses this by sending any number $f(x)$ back to x. Thus the inputs for f^{-1} are the outputs $f(x)$ of f. Think of f^{-1} as a rule that works this way: Given an input value $f(x)$, move horizontally from this point on the y-axis to the graph of f. Then go vertically to the reach the output x on the x-axis.



But this process doesn't work for just any f. Take the f on the right. Starting at $f(x)$ on the y-axis, we can move horizontally to *two* points on the graph, then down to *two* values of x. How can we say which x is the right output for the input $f(x)$? The problem is due to the horizontal line through $f(x)$ that touches the graph of f twice.



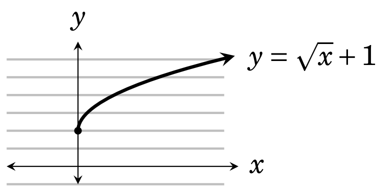
The above discussion means that a function f will have an inverse provided that no horizontal line meets its graph at more than one point. A function that meets this requirement is called **one-to-one**.

4.1 One-to-one Functions and Their Inverses

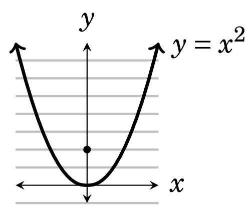
Now we will formalize the previous page's discussion. In order to define the inverse of a function, we first require that it be one-to-one.

Definition 4.1 A function $f(x)$ is **one-to-one** if no horizontal line meets its graph at more than one point.

For example, the function $f(x) = \sqrt{x}+1$ is one-to-one because no horizontal line intersects its graph more than once. (Each horizontal line touches the graph at one point or none at all.)



But the function $f(x) = x^2$ is *not* one-to-one because there exist horizontal lines intersecting the graph at more than one point.



The term *"one-to-one"* comes from the fact that any *one* output number $f(x)$ corresponds to exactly *one* (and not more than one) input number x.

With one-to-one functions defined, we can now define the inverse of such a function. On the previous page we said the inverse of f should be a function f^{-1} sending $f(x)$ back to x, which means $f^{-1}\big(f(x)\big) = x$.

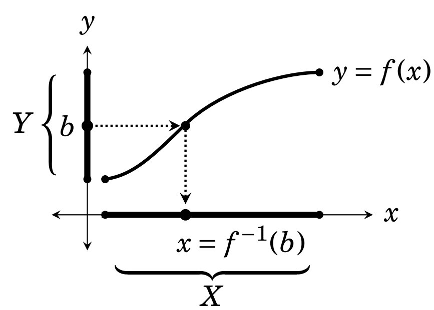
Definition 4.2 Any one-to-one function f with domain X and range Y has an **inverse**, which is a function f^{-1} with domain Y and range X, for which $f^{-1}\big(f(x)\big) = x$ for any x in X, and $f\big(f^{-1}(y)\big) = y$ for any y in Y.

As an illustration, the diagram below shows the graph of a one-to-one function f with domain X and range Y.

The inverse f^{-1} sends any b in Y to the number x in X for which $f(x) = b$. Thus

$$f^{-1}(b) = \left(\begin{array}{c} \text{the number } x \\ \text{for which } f(x) = b \end{array} \right).$$

Thus $f\big(f^{-1}(b)\big) = b$. This and $f^{-1}\big(f(x)\big) = x$ mean that f and f^{-1} "undo" one another.



Above we have $b = f(x)$ and $f^{-1}(b) = x$. In general, if x and y are two numbers for which $y = f(x)$, then $f^{-1}(y) = x$. In other words, the equations $y = f(x)$ and $f^{-1}(y) = x$ express the same relationship between x and y.

On the previous page we encountered the equation

$$f^{-1}(b) = \left(\begin{array}{c} \text{the number } x \\ \text{for which } f(x) = b \end{array} \right). \tag{4.1}$$

This is a rule for f^{-1}. It says that when you are dealing with a number b and are trying to find $f^{-1}(b)$, ask yourself what number x you'd have to plug into f to get b. Then $f^{-1}(b)$ equals that number x. This kind of backwards thinking can often lead to an easy answer.

Example 4.1 Let $f(x) = 2 + x + 2^x$. This function increases as x increases, so it is one-to-one and thus has an inverse. Find $f^{-1}(3)$, $f^{-1}(5)$ and $f^{-1}(8)$.

Let's start with $f^{-1}(3)$. Equation (4.1) says $f^{-1}(3) = \left(\begin{array}{c} \text{the number } x \\ \text{for which } f(x) = 3 \end{array} \right)$. Try plugging a few values into f. You will soon hit upon $f(0) = 2 + 0 + 2^0 = 3$. Thus $f^{-1}(3) = 0$. Also $f(1) = 2 + 1 + 2^1 = 5$, so $f^{-1}(5) = 1$. We are on a roll: $f(2) = 2 + 2 + 2^2 = 8$, so $f^{-1}(8) = 2$.

But computing, say, $f^{-1}(7)$ is not so easy because there is no obvious x for which $f(x) = 7$. This is not to say that $f^{-1}(7)$ doesn't exist (it does). It's just not something we can calculate mentally. ✍

To close, we remark that the function f^{-1} is pronounced "f *inverse*." Please note that f^{-1} is the *symbol* for the function defined by Definition 4.2. It is *not* the reciprocal of f. That is, $f^{-1}(x) \neq \frac{1}{f(x)}$. If we ever wanted to express the reciprocal of $f(x)$ we would write $\big(f(x)\big)^{-1} = \frac{1}{f(x)}$.

And finally, a question. Does the function f^{-1} have an inverse? The equation $f\big(f^{-1}(y)\big) = y$ means that f undoes the effect of f^{-1}, so f is the inverse of f^{-1}. In other words $\big(f^{-1}\big)^{-1} = f$.

Exercises for Section 4.1

1. Suppose $f(x) = 2^x$. Find $f^{-1}(8)$, $f^{-1}(4)$, $f^{-1}(2)$, $f^{-1}(1)$, and $f^{-1}(0.5)$.

2. Suppose $f(x) = 10^x$. Find $f^{-1}(1000)$, $f^{-1}(100)$, $f^{-1}(10)$, $f^{-1}(1)$, and $f^{-1}(0.1)$.

3. Suppose $f(x) = x + x^3$. Find $f^{-1}(2)$, $f^{-1}(10)$, $f^{-1}(-2)$, $f^{-1}(0)$, and $f^{-1}(\sqrt[3]{3} + 3)$.

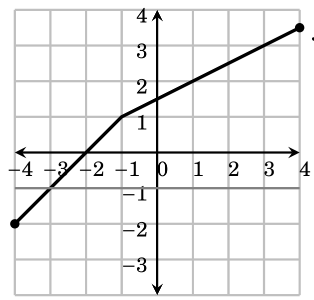
4. Suppose $f(x) = 3^x + x^3$. Find $f^{-1}(4)$, $f^{-1}(17)$, $f^{-1}(1)$, $f^{-1}(54)$ and $f^{-1}(-2/3)$.

5. Suppose $f(x) = x + \sin(x)$. Find $f^{-1}(0)$, $f^{-1}(\pi)$, $f^{-1}(\pi/2 + 1)$ and $f^{-1}(2\pi)$.

6. Suppose $f(x) = x + \cos(x)$. Find $f^{-1}(1)$, $f^{-1}(\pi/2)$, $f^{-1}(\pi - 1)$ and $f^{-1}(2\pi + 1)$.

4.2 Graphing the Inverse of a Function

We now explore how the graph of f^{-1} is related to the graph of f. As starting point consider the one-to-one function f graphed below.



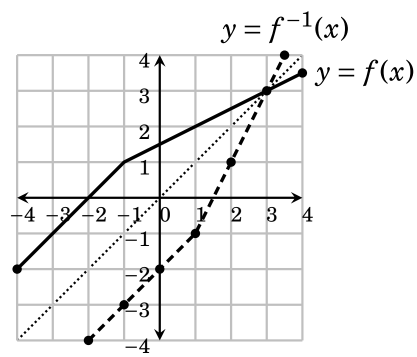
Let's draw the graph of f^{-1}. The range of f is the interval $[-2, 3.5]$, so this is the domain of f^{-1}. To graph f^{-1} we will pick some values in this interval, plug them into f^{-1}, make a table, and sketch the graph.

We learned how to find $f^{-1}(b)$ in Section 4.1. For example, $f^{-1}(2)$ is the number x for which $f(x) = 2$. The graph of f shows $f(1) = 2$, so $f^{-1}(2) = 1$. Similarly $f(4) = 3.5$, so $f^{-1}(3.5) = 4$. Continuing, we get

$$f^{-1}(-2) = -4, \quad f^{-1}(-1) = -3, \quad f^{-1}(0) = -2, \quad f^{-1}(1) = -1, \quad f^{-1}(3) = 3.$$

Next we tally these values in a table, plot the points and connect them. For comparison we show the graph of f^{-1} (dashed) with that of f (solid).

x	$f^{-1}(x)$
−2	−4
−1	−3
0	−2
1	−1
2	1
3	3
3.5	4



There is a striking relationship between the graphs of f and f^{-1}. The graph of f^{-1} is the graph of f reflected across the dotted line. This line is the graph of the equation $y = x$. (Think of it as the graph of the equation $y = 1 \cdot x + 0$, so it is a line with slope 1 and y-intercept 0.)

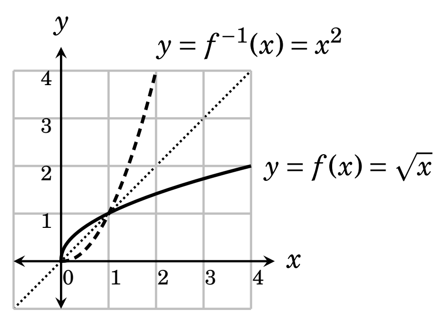
It turns out that the graph of f^{-1} is always the graph of f reflected across the line $y = x$. Before justifying this we look at one more example.

Example 4.2 Consider the function $f(x) = \sqrt{x}$, with domain $[0,\infty)$ and range $[0,\infty)$. Then its inverse also has domain and range $[0,\infty)$. Moreover,

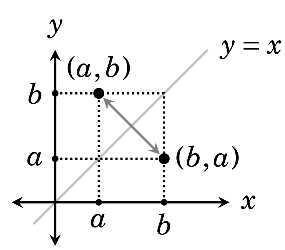
$$f^{-1}(b) = \begin{pmatrix} \text{the number } x \text{ for} \\ \text{which } f(x) = b \end{pmatrix} = \begin{pmatrix} \text{the number } x \text{ for} \\ \text{which } \sqrt{x} = b \end{pmatrix} = b^2.$$

Thus the inverse of $f(x) = \sqrt{x}$ is the function $f^{-1}(x) = x^2$.

The graphs of $y = \sqrt{x}$ and its inverse $y = x^2$ are sketched below. Notice that because the domain of f^{-1} is $[0,\infty)$, the graph of $f^{-1}(x) = x^2$ is plotted only on this domain. (The inverse f^{-1} has no negative inputs because $f(x) = \sqrt{x}$ has no negative outputs.) Just as in the previous example, the graph of f^{-1} is the graph of f reflected across the line $y = x$. ✍



To see why the graph of f^{-1} is always the graph of f reflected across the line $y = x$, recall that the graph of $y = f(x)$ consists of all points $(x, f(x))$ where x is in the domain of f. Because the inverse f^{-1} sends each number $f(x)$ to x, its graph consists of the points $(f(x), x)$. This amounts to saying that for any point (a,b) on the graph of f there is a corresponding point (b,a) on the graph of f^{-1}.

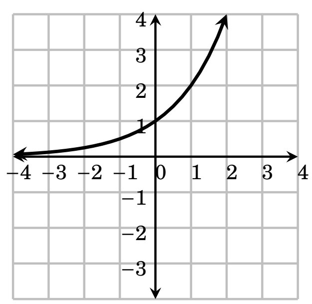


Now think about how points (a,b) and (b,a) are related. The picture above shows that any point (a,b) on the graph of f reflects across the line $y = x$ to (b,a), which is on the graph of f^{-1}. We have established the following fact.

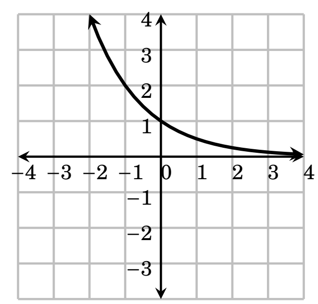
Fact: The graph of f^{-1} is the graph of f reflected across the line $y = x$.

Exercises for Section 4.2

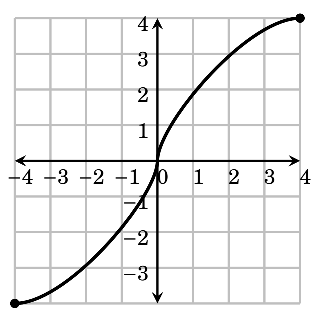
1. Below is the graph of $f(x) = 2^x$.
Find $f^{-1}(4)$, $f^{-1}(2)$, $f^{-1}(1)$, $f^{-1}(1/2)$, $f^{-1}(1/4)$, $f^{-1}(1/8)$, and $f^{-1}(1/16)$.
Sketch the graph of f^{-1}.



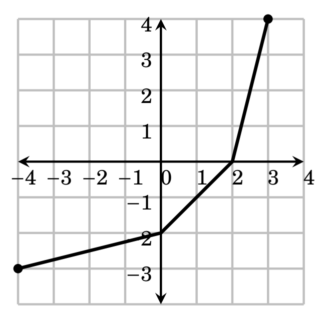
2. Below is the graph of $f(x) = \dfrac{1}{2^x}$.
Find $f^{-1}(4)$, $f^{-1}(2)$, $f^{-1}(1)$, $f^{-1}(1/2)$, $f^{-1}(1/4)$, $f^{-1}(1/8)$, and $f^{-1}(1/16)$.
Sketch the graph of f^{-1}.



3. A function f is graphed below.
Find $f^{-1}(-4)$, $f^{-1}(-3)$, $f^{-1}(-2)$, $f^{-1}(0)$, $f^{-1}(2)$, $f^{-1}(3)$ and $f^{-1}(4)$.
Sketch the graph of f^{-1}.



4. A function f is graphed below.
Find $f^{-1}(-3)$, $f^{-1}(-2)$, $f^{-1}(-1)$, $f^{-1}(0)$, $f^{-1}(2)$, and $f^{-1}(4)$.
Sketch the graph of f^{-1}.



5. Here is a table for some values of a one-to-one function f. Use it to make a table for f^{-1}. Sketch the graphs of f and f^{-1}.

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$	5	4	3	2	1	0.75	0.5	0.25	0

6. Here is a table for some values of a one-to-one function f. Use it to make a table for f^{-1}. Sketch the graphs of f and f^{-1}.

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$	−4	−3.5	−3	−2.5	−1	1	3	3.5	5

4.3 Finding Inverses

This section reviews a technique for finding the function $f^{-1}(x)$ when $f(x)$ is given as an algebraic expression.

As a point of departure we note one of many uses for inverses: they can be used to solve certain equations. For example, suppose we have an equation of form

$$y = f(x)$$

and we want to solve it for x in terms of y. That is, we want to isolate the x on one side of the equation, and have on the other side an expression involving the variable y. If f happens to have an inverse, then we can take f^{-1} of both sides of the above equation to get $f^{-1}(y) = f^{-1}\big(f(x)\big)$. As Definition 4.2 assures us that $f^{-1}\big(f(x)\big) = x$, this becomes

$$f^{-1}(y) = x,$$

and we have now solved for x in terms of y.

In summary, solving $y = f(x)$ for x yields $x = f^{-1}(y)$.

Now suppose we know the function $f(x)$ but do not know what $f^{-1}(x)$ is. The above discussion tells us that we can find $f^{-1}(x)$ by algebraically solving the equation $y = f(x)$ for x, and we will get $x = f^{-1}(y)$. Of course we will probably want to use x as the independent variable and y as a dependent variable, so we can interchange them to get $y = f^{-1}(x)$, and the inverse is at hand. Here is a summary of our technique.

How to compute the inverse of $f(x)$

1. Write $y = f(x)$
2. Interchange x and y to get $x = f(y)$.
3. Solve $x = f(y)$ for y to get $y = f^{-1}(x)$.

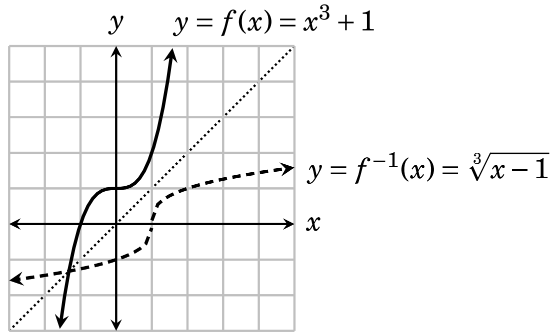
Example 4.3 Find the inverse of the function $f(x) = x^3 + 1$.

Let us carry out the steps of our new procedure.

1. Write $y = f(x)$, which in this case is $y = x^3 + 1$.
2. Interchange x and y to get $x = y^3 + 1$.
3. Next we solve $x = y^3 + 1$ for y.
   $x - 1 = y^3$
   $\sqrt[3]{x-1} = y$.
   Therefore $y = \sqrt[3]{x-1}$, and we conclude $f^{-1}(x) = \sqrt[3]{x-1}$.

The inverse has now been computed and it is $f^{-1}(x) = \sqrt[3]{x-1}$. ✍

Let's round out Example 4.3 by comparing the graphs of $f(x)$ and $f^{-1}(x)$. This is a good opportunity to use the graph-shifting techniques of Section 2.4. the graph of $f(x) = x^3 + 1$ is the graph of $y = x^3$ shifted up one unit, and the graph of $f^{-1}(x) = \sqrt[3]{x-1}$ is the graph of $y = \sqrt[3]{x}$ shifted right one unit. The shifted graphs are sketched below. As expected, one is the reflection of the other across the line $y = x$.



If you are ever in doubt that the inverse you have computed is correct, there is a way to check it. Definition 4.2 says both equations $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(x)\big) = x$ must hold. Verifying one (or both) of these assures you that your work is correct. In the example just done, we started with $f(x) = x^3 + 1$ and obtained $f^{-1}(x) = \sqrt[3]{x-1}$. Note that

$$\begin{aligned} f^{-1}\big(f(x)\big) &= \sqrt[3]{f(x)-1} \\ &= \sqrt[3]{x^3+1-1} \\ &= \sqrt[3]{x^3} \\ &= x. \end{aligned}$$

The fact that $f^{-1}\big(f(x)\big) = x$ indicates that our work was correct. The inverse f^{-1} literally "undoes" the effect of f, sending any number $f(x)$ back to x.

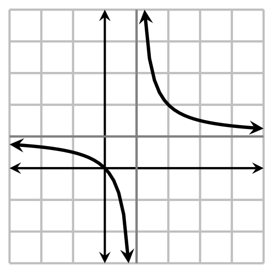
This section has used only the letter f for a function. Of course other letters can be used. A one-to-one function g will have an inverse g^{-1}, etc.

Example 4.4 Is the function $g(x) = \dfrac{x}{x-1}$ one-to-one? If so, find its inverse.

To answer the first question, let's sketch the graph of g and see whether any horizontal line crosses its graph more than once. To draw the graph we can manipulate g slightly and apply shifting. Notice that

$$g(x) = \frac{x}{x-1} = \frac{1+(x-1)}{x-1} = \frac{1}{x-1} + \frac{x-1}{x-1} = \frac{1}{x-1} + 1.$$

As $g(x) = \dfrac{1}{x-1} + 1$, we see that its graph is the graph of $y = \dfrac{1}{x}$ shifted right one unit and up one unit. This is sketched below. As no horizontal line crosses the graph more than once, g is one-to-one, so it has an inverse.



Now we carry out our procedure for computing g^{-1} from g. The first step is to write the equation $y = g(x)$, which in this case is

$$y = \frac{x}{x-1}.$$

Next we interchange x and y to get

$$x = \frac{y}{y-1}.$$

Now we must solve this for y. We want to isolate y, so as a fist step we get the y out of the denominator by multiplying both sides by $y-1$.

$$\begin{aligned} x(y-1) &= \frac{y}{y-1}(y-1) \\ xy - x &= y \end{aligned}$$

Now we need to collect all occurrences of y on one side. Doing this, we get

$$xy - y = x.$$

To isolate y we factor it out on the left and divide both sides by $x-1$.

$$\begin{aligned} y(x-1) &= x \\ \frac{y(x-1)}{x-1} &= \frac{x}{x-1} \\ y &= \frac{x}{x-1} \end{aligned}$$

Now that the equation has been solved for y we find that $g^{-1}(x) = \dfrac{x}{x-1}$. In summary, the inverse of $g(x) = \dfrac{x}{x-1}$ is the function $g^{-1}(x) = \dfrac{x}{x-1}$.

Interestingly, the inverse of g is g itself, that is, g is its own inverse. This is just a coincidence – most functions are not equal to their inverses. But perhaps we should not have been surprised that this g is its own inverse: Looking at the above graph of g, we see that it is symmetric with respect to the line $y = x$, that is, reflecting it across the line does not yield a new graph. Thus the graphs of g and g^{-1} are identical, so they are the same function. ✍

Our technique of finding the inverse of f by solving $x = f(y)$ for y has its limitations because the equation may be difficult or impossible to solve.

Consider the function $f(x) = x + \sin(x)$ from Exercise 5 on page 61. By "reverse engineering" we can compute $f^{-1}(x)$ for certain convenient values of x. For instance, if asked about $f^{-1}(\pi/2+1)$ we would (after some thought) note that $f(\pi/2) = \pi/2 + \sin(\pi/2) = \pi/2 + 1$ and therefore $f^{-1}(\pi/2+1) = \pi/2$.

But actually finding a formula for $f^{-1}(x)$ is problematic. It involves solving the equation $x = y + \sin(y)$ for y. This is a very difficult problem because our standard equation solving techniques cannot isolate the y.

Exercises for Section 4.3

Each of the following functions is one-to-one. Use the methods of this section to find their inverses.

1. $f(x) = (x-3)^3 - 1$

2. $f(x) = -\dfrac{1}{x}$

3. $g(x) = -\sqrt[5]{x+2}$

4. $f(x) = 2 - x^3$

5. $f(x) = \dfrac{2-x}{x+5}$

6. $f(x) = \dfrac{2-3x}{x+5}$

7. $f(x) = x^3 + 3x^2 + 3x + 1$
(Hint: factor first.)

8. $f(x) = x^3 + 3x^2 + 3x$
(Hint: compare to Exercise 7.)

9. $f(x) = 2 - x^2$ on domain $[0, \infty)$

10. $h(x) = \dfrac{2}{\sqrt[3]{x}}$

11. $f(\theta) = \dfrac{1}{\theta + 3}$

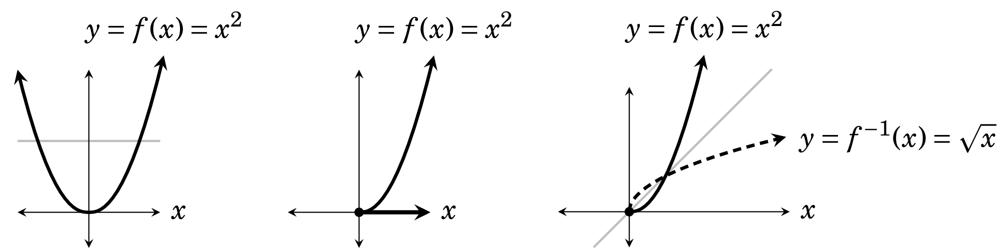
12. $f(x) = 1 - \dfrac{1}{x}$

13. $g(x) = 3 - 5\sqrt[3]{4x-3}$

14. $f(w) = \dfrac{1}{w^3 + 3}$

4.4 Restricting the Domain

If a function is not one-to-one, then it has no inverse. This is the case for $f(x) = x^2$ graphed below, left. A horizontal line crosses its graph twice, so f is not one-to-one and thus has no inverse.

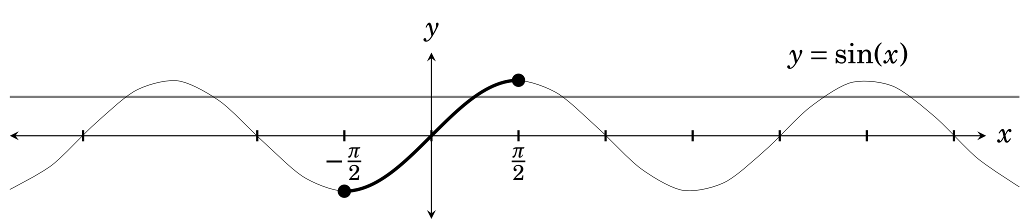


Occasionally we will be in a situation where it is still desirable to find an "inverse" of such a function. This task is not as hopeless as it may seem. One approach is to restrict the domain of f to make it one-to-one.

For the function $f(x) = x^2$, we could declare the domain to be the interval $[0,\infty)$, rather than its natural domain $(-\infty,\infty)$. This middle graph above shows $y = f(x)$ with this domain. Now f *is* one-to-one, and indeed it has an inverse $f^{-1}(x) = \sqrt{x}$. The right-hand graph shows f with this *restricted domain*, along with its inverse $f^{-1}(x) = \sqrt{x}$.

Modifying the domain of a function to make it one-to-one is called *restricting the domain*. We will rarely have to use this technique, so it is not necessary to work any exercises involving it. It is used only in Chapter 6 where we develop the useful idea of inverse trig functions.

A function like $\sin(x)$ is clearly not-one-to one because many horizontal lines cross it infinitely many times. But if we restrict its domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the resulting function (graphed bold below) is one-to-one and has an inverse.



Chapter 6 will develop this idea. We will restrict the domains of the trigonometric functions to make them one-to-one. We will then be able to define such functions as $\sin^{-1}(x)$, $\tan^{-1}(x)$, etc.

4.5 Exercise Solutions for Chapter 4

Exercises for Section 4.1

1. Suppose $f(x) = 2^x$. Find $f^{-1}(8)$, $f^{-1}(4)$, $f^{-1}(2)$, $f^{-1}(1)$, and $f^{-1}(0.5)$.

$f(3) = 8$, so $\boxed{f^{-1}(8) = 3}$ $f(2) = 4$, so $\boxed{f^{-1}(4) = 2}$ $f(1) = 2$, so $\boxed{f^{-1}(2) = 1}$

$f(-1) = 2^{-1} = \frac{1}{2} = 0.5$, so $\boxed{f^{-1}(0.5) = -1}$

3. Suppose $f(x) = x + x^3$. Find $f^{-1}(2)$, $f^{-1}(10)$, $f^{-1}(-2)$, $f^{-1}(0)$, and $f^{-1}(\sqrt[3]{3} + 3)$.

$f(1) = 2$, so $\boxed{f^{-1}(2) = 1}$ $f(2) = 10$, so $\boxed{f^{-1}(10) = 2}$ $f(-1) = -2$, so $\boxed{f^{-1}(-2) = -1}$

$f(\sqrt[3]{3}) = \sqrt[3]{3} + \sqrt[3]{3}^3 = \sqrt[3]{3} + 3$, so $\boxed{f^{-1}(\sqrt[3]{3} + 3) = \sqrt[3]{3}}$

5. Suppose $f(x) = x + \sin(x)$. Find $f^{-1}(0)$, $f^{-1}(\pi)$, $f^{-1}(\pi/2 + 1)$ and $f^{-1}(2\pi)$.

$f(0) = 0$, so $\boxed{f^{-1}(0) = 0}$ $f(\pi) = \pi$, so $\boxed{f^{-1}(\pi) = \pi}$

$f(\pi/2) = \pi/2 + 1$, so $\boxed{f^{-1}(\pi/2 + 1) = \pi/2}$ $f(2\pi) = 2\pi$, so $\boxed{f^{-1}(2\pi) = 2\pi}$

Exercises for Section 4.2

1. Below is the graph of $f(x) = 2^x$.

$f(2) = 4$, so $f^{-1}(4) = 2$

$f(1) = 2$, so $f^{-1}(2) = 1$

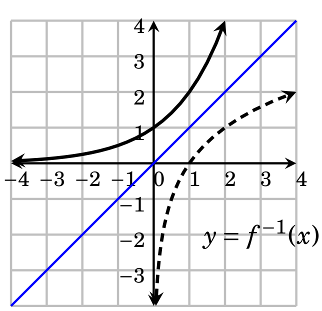
$f(0) = 1$, so $f^{-1}(1) = 0$

$f(-1) = 1/2$, so $f^{-1}(1/2) = -1$,

$f(-2) = 1/4$, so $f^{-1}(1/4) = -2$

$f(-3) = 1/8$, so $f^{-1}(1/8) = -3$

$f(-4) = 1/16$, so $f^{-1}(1/16) = -4$.



2. A function f is graphed below.

$f(-4) = -4$, so $f^{-1}(-4) = -4$

$f(-2) = -3$, so $f^{-1}(-3) = -2$

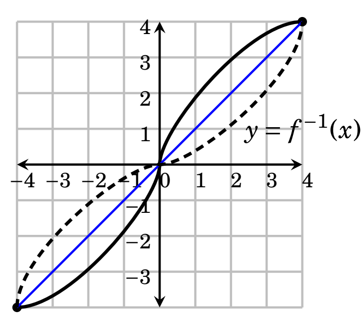
$f(-1) = -2$, so $f^{-1}(-2) = -1$

$f(0) = -0$, so $f^{-1}(0) = 0$

$f(1) = 2$, so $f^{-1}(2) = 1$

$f(2) = 3$, so $f^{-1}(3) = 2$

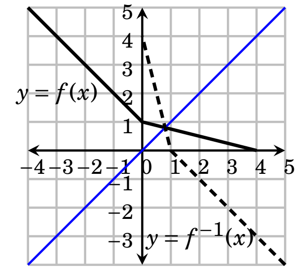
$f(4) = 4$, so $f^{-1}(4) = 4$



3. Here is a table for some values of a one-to-one function f. Use it to make a table for f^{-1}. Sketch the graphs of f and f^{-1}.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	5	4	3	2	1	0.75	0.5	0.25	0

x	5	4	3	2	1	0.75	0.5	0.25	0
$f^{-1}(x)$	-4	-3	-2	-1	0	1	2	3	4



Exercises for Section 4.3

1. Find the inverse: $f(x) = (x-3)^3 - 1$

$$\begin{aligned} y &= (x-3)^3 - 1 \\ x &= (y-3)^3 - 1 \\ x+1 &= (y-3)^3 \\ \sqrt[3]{x+1} &= y-3 \\ y &= \sqrt[3]{x+1}+3 \\ f^{-1}(x) &= \sqrt[3]{x+1}+3 \end{aligned}$$

3. Find the inverse: $g(x) = -\sqrt[5]{x+2}$

$$\begin{aligned} y &= -\sqrt[5]{x+2} \\ x &= -\sqrt[5]{y+2} \\ x^5 &= \left(-\sqrt[5]{y+2}\right)^5 \\ x^5 &= -(y+2) \\ y &= -2-x^5 \\ g^{-1}(x) &= -2-x^5 \end{aligned}$$

5. Find the inverse: $f(x) = \dfrac{2-x}{x+5}$

$$\begin{aligned} y &= \frac{2-x}{x+5} \\ x &= \frac{2-y}{y+5} \\ x(y+5) &= 2-y \\ xy+5x &= 2-y \\ xy+y &= 2-5x \\ y(x+1) &= 2-5x \\ y &= \frac{2-5x}{x+1} \\ f^{-1}(x) &= \frac{2-5x}{x+1} \end{aligned}$$

7. Find the inverse: $f(x) = x^3+3x^2+3x+1$

$$\begin{aligned} y &= x^3+3x^2+3x+1 \\ x &= y^3+3y^2+3y+1 \\ x &= (y+1)^3 \\ \sqrt[3]{x} &= \sqrt[3]{(y+1)^3} \\ \sqrt[3]{x} &= y+1 \\ y &= \sqrt[3]{x}-1 \\ f^{-1}(x) &= \sqrt[3]{x}-1 \end{aligned}$$

9. Find the inverse: $f(x) = 2-x^2$

$$\begin{aligned} y &= 2-x^2 \\ x &= 2-y^2 \\ y^2 &= 2-x \\ y &= \sqrt{2-x} \\ f^{-1}(x) &= \sqrt{2-x} \end{aligned}$$

11. Find the inverse of $f(\theta) = \dfrac{1}{\theta+3}$

$$
\begin{aligned}
y &= \frac{1}{\theta+3} \\
\theta &= \frac{1}{y+3} \\
\theta(y+3) &= 1 \\
\theta y + 3\theta &= 1 \\
\theta y &= 1-3\theta \\
y &= \frac{1-3\theta}{\theta} \\
f^{-1}(\theta) &= \frac{1-3\theta}{\theta}
\end{aligned}
$$

13. Find the inverse: $g(x) = 3 - 5\sqrt[3]{4x-3}$

$$
\begin{aligned}
y &= 3-5\sqrt[3]{4x-3} \\
x &= 3-5\sqrt[3]{4y-3} \\
3-x &= 5\sqrt[3]{4y-3} \\
\frac{3-x}{5} &= \sqrt[3]{4y-3} \\
\left(\frac{3-x}{5}\right)^3 &= 4y-3 \\
\left(\frac{3-x}{5}\right)^3 + 3 &= 4y \\
y &= \frac{1}{4}\left(\frac{3-x}{5}\right)^3 + \frac{3}{4} \\
g^{-1}(x) &= \frac{1}{4}\left(\frac{3-x}{5}\right)^3 + \frac{3}{4}
\end{aligned}
$$