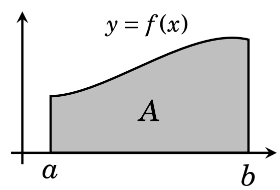
CHAPTER 40

Area

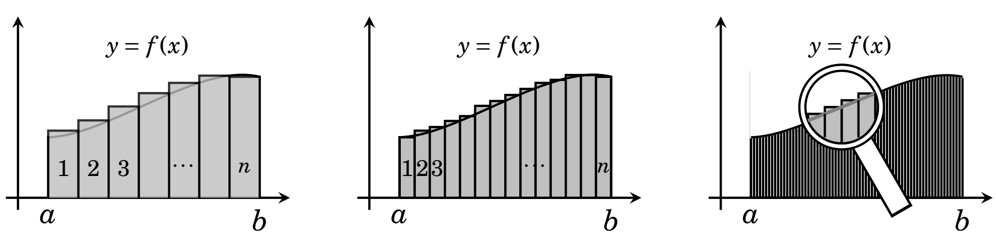
The two main themes of this course have been differentiation and its opposite process, integration. Beginning in this chapter, and continuing through the remainder of the text, we explore a startling connection between these two processes. It is related to the area under the graph of a function $y = f(x)$ and between two x values a and b



This connection is called the Fundamental Theorem of Calculus, and is covered in Chapter 42. As we will see, one consequence of this theorem is a simple formula for the area under a curve in terms an antiderivative of f.

In preparation for this, we now consider the problem of finding the area under a function. Our investigations will lead to a major definition called the *definite integral*, which will be the subject of Chapter 41. Then we will be ready for the fundamental theorem of calculus, in Chapter 42.

Since the region under a curve is unlikely to be a geometric shape that has a formula for its area, we will instead express its area by approximating it with smaller shapes whose area we *can* compute, namely rectangles.



Our approach will be to approximate the region with some number n of rectangular strips, reaching from the x-axis to the curve, as suggested in the diagrams above. Then the area A we seek is approximated by the sum of the areas (base×height) of the n rectangles. The more rectangles we use, the better they fit the contour of the curve, and

$$A = \lim_{n\to\infty} \left(\text{sum of the areas of } n \text{ rectangles}\right)$$

If this limit turns out to be one that we can compute, then we have a formula for A. Even if the limit is not one that can be easily found, this theoretical approach to A will still turn out to be highly productive.

However, this approach involves sums with very many terms, because the number of rectangles approaches infinity. Before making further progress with this approach we need a notation that effectively handles such large sums. That notation is called *sigma notation*.

40.1 Sigma Notation

In mathematics, the upper case greek letter Σ (sigma) is commonly used to indicate a sum. This is done as follows.

Sigma Notation

If f is a function, n is a positive integer, and k is a variable, then

$$\sum_{k=1}^{n} f(k) = f(1)+f(2)+f(3)+f(4)+\cdots+f(n).$$

We read $\sum_{k=1}^{n} f(k)$ as "*sum of $f(k)$ from 1 to n*". It means to add up the terms $f(1)+f(2)+f(3)+f(4)+\cdots$, stopping when you get to the nth term $f(n)$.

For example, Suppose f is the function defined as $f(k)=k^2+1$. Then:

$$\begin{aligned}
\sum_{k=1}^{2}(k^2+1) &= (1^2+1)+(2^2+1) = \mathbf{7}\\
\sum_{k=1}^{3}(k^2+1) &= (1^2+1)+(2^2+1)+(3^2+1) = \mathbf{17}\\
\sum_{k=1}^{4}(k^2+1) &= (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1) = \mathbf{34}\\
\sum_{k=1}^{5}(k^2+1) &= (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)+(5^2+1) = \mathbf{60}
\end{aligned}$$

For another example, let $f(k)=k$ be the identity function. Then

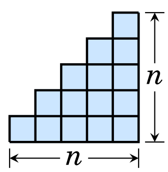
$$\sum_{k=1}^{10} k = 1+2+3+4+5+6+7+8+9+10 = \mathbf{55}.$$

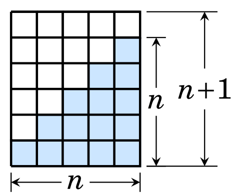
Sometimes you will encounter a sum like $\sum_{k=1}^{20} 3$. For something like this, the 3 should be interpreted as the constant function $f(k)=3$, so

$$\sum_{k=1}^{20} 3 \;=\; \underbrace{3+3+3+3+3+\cdots+3}_{\text{20 times}} \;=\; \mathbf{60}.$$

Thus, in general, $\sum_{k=1}^{n} c = nc$, which can be regarded as a *formula* for the sum. For example, $\sum_{k=1}^{1000} 23 = 23 \cdot 1000 = 23000$.

Another basic sum that has a simple formula is $\sum_{k=1}^{n} k = 1+2+3+4+\cdots+n$. Interpret this sum as the area of a stair-step shaped region n units wide by n units high as shown below, left. (The first column of squares has area 1, the second has area 2, and so on.) The area of this region is $\sum_{k=1}^{n} k$.





Put together two copies of this region and you get an $n \times (n+1)$ rectangle (above, right) with area $n(n+1)$. Therefore $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. For example,

$$\sum_{k=1}^{10} k \;=\; \frac{10(10+1)}{2} \;=\; \frac{110}{2} \;=\; \mathbf{55},$$

the same result we got by regular addition on the previous page.

Here is a list of the above sum formulas, and two others.

Fact 40.1 (Sum Formulas)

1. $\displaystyle\sum_{k=1}^{n} c \;=\; c+c+c+\cdots+c \;=\; nc$
2. $\displaystyle\sum_{k=1}^{n} k \;=\; 1+2+3+\cdots+n \;=\; \frac{n(n+1)}{2}$
3. $\displaystyle\sum_{k=1}^{n} k^2 \;=\; 1^2+2^2+3^2+\cdots+n^2 \;=\; \frac{n(n+1)(2n+1)}{6} \;=\; \frac{2n^3+3n^2+n}{6}$
4. $\displaystyle\sum_{k=1}^{n} k^3 \;=\; 1^3+2^3+3^3+\cdots+n^3 \;=\; \frac{n^2(n+1)^2}{4}$

We will not concern ourselves here with verifying the above formulas 3 and 4. In fact, these formulas will be used only in this section, and do not play a major role in Calculus I.

More important are the following three properties of sums, which are mere restatements of the associative and distributive properties of addition.

Fact 40.2 (Sum Rules)

1. $\sum_{k=1}^{n}\Big(f(k)+g(k)\Big) = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)$
2. $\sum_{k=1}^{n}\Big(f(k)-g(k)\Big) = \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n} g(k)$
3. $\sum_{k=1}^{n} c\,f(k) = c\cdot\sum_{k=1}^{n} f(k)$

Many sums follow quickly and easily from these rules and formulas.

Example 40.1

$$\begin{aligned}
\sum_{k=1}^{100}\left(\frac{1}{3}k^2+4k-2\right) &= \sum_{k=1}^{100}\frac{1}{3}k^2 + \sum_{k=1}^{100}4k - \sum_{k=1}^{100}2 && \text{(Rules 1,2)}\\
&= \frac{1}{3}\sum_{k=1}^{100}k^2 + 4\sum_{k=1}^{100}k - \sum_{k=1}^{100}2 && \text{(Rule 3)}\\
&= \frac{1}{3}\cdot\frac{100(100+1)(2\cdot100+1)}{6} + 4\cdot\frac{100(100+1)}{2} - 100\cdot 2 && \text{(Formulas 3, 2, 1)}\\
&= \frac{2030100}{18} + \frac{40400}{2} - 200\\
&= \frac{338350}{3} + 20200 - 200\\
&= \frac{338350}{3} + \frac{60600}{3} - \frac{600}{3} = \frac{398350}{3}
\end{aligned}$$

✍

Two final remarks about sigma notation are in order. First, the starting value of k need not be 1. It could be any integer up to n. For example,

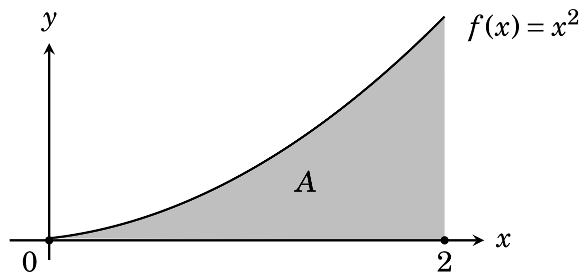
$$\sum_{k=5}^{10} f(k) = f(5)+f(6)+f(7)+f(8)+f(9)+f(10).$$

(But note that Formulas 1, 2 and 3 require the initial value $k=1$.) Second, variables other than k are admissible. For example, $\sum_{k=1}^{n} f(k) = \sum_{i=1}^{n} f(i)$. The variables k or i in the sums are sometimes called *dummy variables* because the sum's value does not depend on which variable is used.

40.2 Area

Now we will use sigma notation to help find area under a curve. We will illustrate this with an example. Our approach will then lead to a general formula for area under the graph of a function.

Example 40.2 Our problem here is to find the area A of the region under the graph of $f(x) = x^2$, between $x = 0$ and $x = 2$ (shown shaded on the right).



To do this we first fix a positive integer n. Next we will cover this region with n rectangular strips.

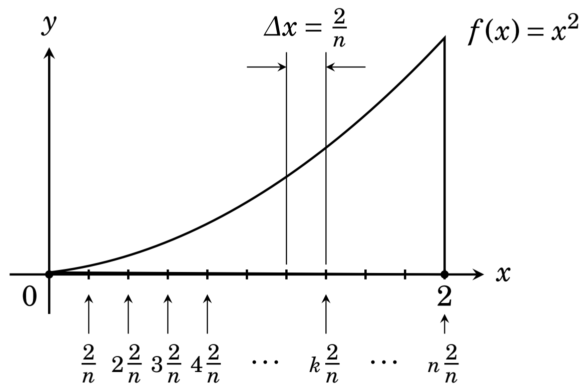
To do this, divide the interval $[0,2]$ (which has length 2) into n pieces (called *subintervals*) of equal length. Call this length Δx, so

$$\Delta x = \frac{2}{n}.$$

The endpoints of the subintervals are

$$0,\ \frac{2}{n},\ 2\frac{2}{n},\ 3\frac{2}{n},\ \ldots,\ n\frac{2}{n} = 2,$$

as indicated on the right.

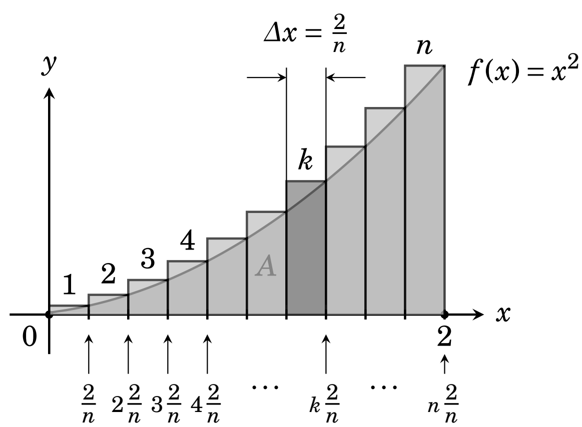


Now let each subinterval be the base of a rectangle whose upper right corner touches the graph of $f(x)$. The kth rectangle (for $1 \le k \le n$) has height $f\left(k\frac{2}{n}\right)$ and base Δx, so its area is

$$\text{height} \times \text{base} \;=\; f\left(k\tfrac{2}{n}\right)\Delta x.$$

The area A is approximately the sum of the areas of these n rectangles:

$$A \approx \sum_{k=1}^{n} f\left(k\frac{2}{n}\right)\Delta x.$$



Let's now work out this sum. Using the fact that $f(x) = x^2$, $\Delta x = \frac{2}{n}$, along with Rule 3 and Formula 3 from the previous section, we get

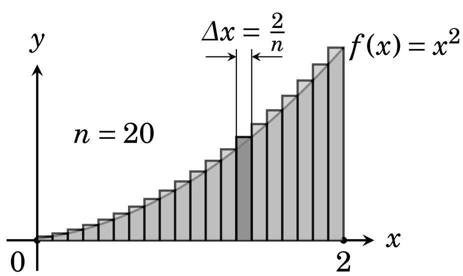
$$\begin{aligned} A \approx \sum_{k=1}^{n} f\left(k\frac{2}{n}\right)\Delta x \;&=\; \sum_{k=1}^{n}\left(k\frac{2}{n}\right)^2\frac{2}{n} \;=\; \sum_{k=1}^{n}\frac{8}{n^3}k^2 \;=\; \frac{8}{n^3}\sum_{k=1}^{n}k^2 \;=\; \frac{8}{n^3}\,\frac{2n^3+3n^2+n}{6} \\ &=\; \frac{4}{3}\,\frac{2n^3+3n^2+n}{n^3}. \end{aligned}$$

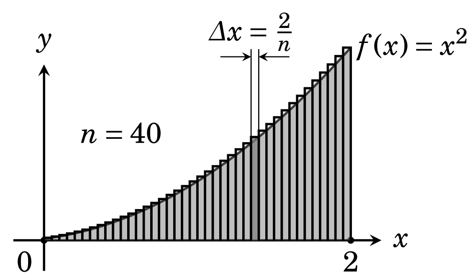
To summarize what we've done so far, the area A we seek is approximated by the sum of the areas of n rectangles, and this sum is

$$A \;\approx\; \sum_{k=1}^{n} f\left(k\frac{2}{n}\right)\Delta x \;=\; \frac{4}{3}\,\frac{2n^3+3n^2+n}{n^3}. \qquad (*)$$

Notice that this value depends on n, the number of rectangles used. Actually, it is slightly larger than A, because the upper-left corners of the rectangles extend outside of the region whose area we want to measure.

To get a better approximation just increase the number n of rectangles, for the more rectangles we use, the better they fit the contour of the curve. The pictures below show 20 and 40 rectangles, respectively.





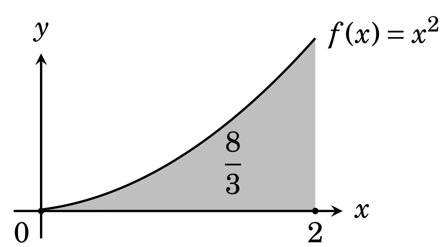
So to get the area A exactly, all we have to do is let the number n of rectangles in $(*)$ approach infinity. Thus:

$$\begin{aligned} A \;&=\; \lim_{n\to\infty}\sum_{k=1}^{n} f\left(k\frac{2}{n}\right)\Delta x \;=\; \lim_{n\to\infty}\frac{4}{3}\,\frac{2n^3+3n^2+n}{n^3} \\ &=\; \frac{4}{3}\cdot 2 \;=\; \frac{8}{3}. \end{aligned}$$

(The limit above has indeterminate form $\frac{\infty}{\infty}$, and is found either with L'Hôpital's rule or with techniques from Chapter 13.)

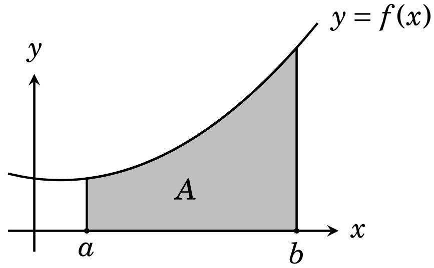
We have now computed the area under the graph of $f(x)$ as a limit of the areas of rectangles.

Answer: The area under the graph of $f(x) = x^2$ and between $x = 0$ and $x = 2$ is $\frac{8}{3}$ square units. ✍



Following the approach of Example 40.2, we now develop a formula for area under a curve.

Take a function $f(x)$ that is positive on an interval $[a,b]$. Our goal is a formula for the area A of the region shown below.



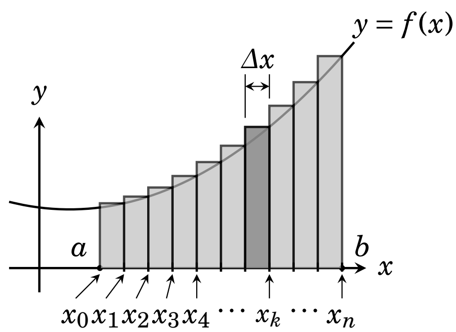
As in Example 40.2, the first step is to cover this region with n rectangles. So start by dividing $[a,b]$ into n subintervals of equal length. As $[a,b]$ has length $b-a$, each subinterval has length

$$\Delta x = \frac{b-a}{n}.$$

Label the endpoints of the subintervals x_0, x_1, x_2, ..., x_n, where

$$x_k = a + k\Delta x.$$

(So $x_0 = a+0\cdot\Delta x = a$ and $x_n = a+n\cdot\Delta x = a+n\frac{b-a}{n} = b$.) Thus the subintervals are $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, ..., $[x_{n-1},x_n]$, and the kth subinterval is $[x_{k-1},x_k]$.



Next, let each subinterval $[x_{k-1},x_k]$ be the base of a rectangle whose height is $f(x_k)$. Then the upper right corner of the kth rectangle is on the graph of $f(x)$, and the kth rectangle (shown shaded above) has area $f(x_k)\Delta x$.

Thus A is approximately equal to the sum of the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} f(x_k)\Delta x.$$

To get A exactly, we let the number n of rectangles approach infinity:

$$\boxed{A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x.}$$

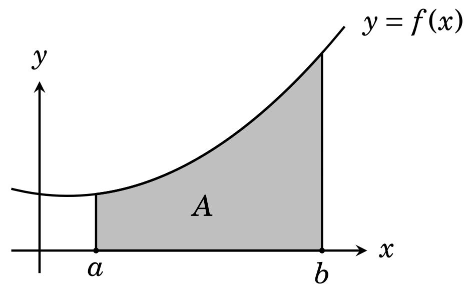
This is our formula for area under a curve.

Fact 40.3 (Area under a curve)

If a function $f(x)$ is continuous and nonnegative on an interval $[a,b]$, then the area of the region below its graph and above this interval is

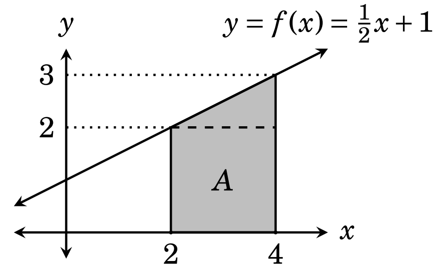
$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_k = a + k\Delta x$.



Example 40.3 Find the area below $y = f(x) = \frac{1}{2}x + 1$ and above $[2,4]$.

Solution This problem is intentionally simple, because $f(x) = \frac{1}{2}x+1$ is a linear function. The region in question is the union of a 2×2 square and a triangle of base 2 and height 1. Therefore its area is $A = 2\cdot 2 + \frac{1}{2}2\cdot 1 = 5$ square units.



But let's test Fact 40.3 by using *it* to get the same answer. To set up the area formula, we need to find Δx and x_k. In this problem the interval is $[a,b] = [2,4]$ so $\Delta x = (4-2)/n = 2/n$. Also $x_k = a + k\Delta x = 2 + k2/n$. Thus

$$\begin{aligned}
A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x &= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(2 + \frac{2k}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{2}\left(2+\frac{2k}{n}\right)+1\right)\frac{2}{n} && (f(x) = \tfrac{1}{2}x+1) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4}{n} + \frac{2k}{n^2}\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{4}{n} + \sum_{k=1}^{n} \frac{2k}{n^2}\right) && \text{(Fact 40.2 (2))} \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{k=1}^{n} 4 + \frac{2}{n^2}\sum_{k=1}^{n} k\right) && \text{(Fact 40.2 (3))} \\
&= \lim_{n\to\infty} \left(\frac{1}{n}4n + \frac{2}{n^2}\frac{n(n+1)}{2}\right) && \text{(Fact 40.1 (1 \& 2))} \\
&= \lim_{n\to\infty} \left(4 + \frac{n^2+n}{n^2}\right) = 4+1 = 5 \text{ square units.}
\end{aligned}$$

✍

Although Fact 40.3 gives area exactly, we will rarely if ever use it. One reason is that the limit can be difficult or cumbersome. But a more important reason is that the fundamental theorem of calculus in Chapter 42 will provide a shortcut method of computing A by means of antiderivatives of f. In this sense the formula for A (Fact 40.3) is akin to the definition of the derivative, $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$. This limit was important because it gave slope. But then we found ways to compute $f'(x)$ without a limit. The same is true for area.

The real importance of Fact 40.3 is that it motivates the idea of a *definite integral*, which is of major importance in calculus. We will define the definite integral in the next chapter.

In short, our area formula (Fact 40.3) is of more theoretical than practical value. Consequently it is not vitally important to compute area using it, as we did in this chapter. However, doing an exercise or two can help drive home the important ideas that will come to bear in the next two chapters.

Exercises for Chapter 40

In Exercises 1–8, use this chapter's sum formulas and rules to find the sums.

1. $\sum\limits_{k=1}^{100} 3$ **2.** $\sum\limits_{k=1}^{100} (3+k)$ **3.** $\sum\limits_{k=1}^{500} k^2$ **4.** $\sum\limits_{k=1}^{10} k^3$

5. $\sum\limits_{k=1}^{5} (3+2k)$ **6.** $\sum\limits_{k=1}^{40} (1+k+k^2)$ **7.** $\sum\limits_{k=1}^{100} (2k^2-4)$ **8.** $\sum\limits_{k=1}^{100} (k^2+2)$

9. Consider the region contained under the graph of $f(x) = x+1$ between $x = 1$ and $x = 2$. Since the graph of f is a straight line, you can compute the area of this region by dividing it into a rectangle and a triangle. Do so. Then arrive at the same answer by using Fact 40.3.

10. Use Fact 40.3 to find the area of the region contained under the graph of $f(x) = x^2+1$ and between $x = 1$ and $x = 3$.

Exercise Solutions for Chapter 40

1. $\sum_{k=1}^{100} 3 = 100 \cdot 3 = 300$ (By Fact 40.1 (1))

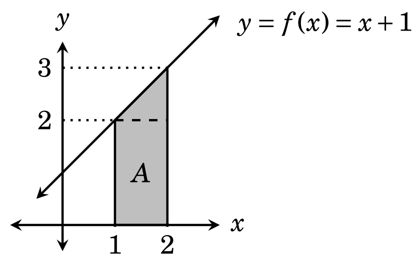
3. $\sum_{k=1}^{500} k^2 = \frac{500(500+1)(2\cdot 500+1)}{6} = \frac{500\cdot 501\cdot 1001}{6} = 41,791,750$ (By Fact 40.1 (3))

5. $\sum_{k=1}^{5}(3+2k) = \sum_{k=1}^{5} 3 + \sum_{k=1}^{5} 2k = \sum_{k=1}^{5} 3 + 2\sum_{k=1}^{5} k = 5\cdot 3 + 2\frac{5(5+1)}{2} = 45$

7. $\sum_{k=1}^{100}(2k^2-4) = \sum_{k=1}^{100} 2k^2 - \sum_{k=1}^{100} 4 = 2\sum_{k=1}^{100} k^2 - 100\cdot 4 = 2\frac{100(100+1)(2\cdot 100+1)}{6} - 400 = 676300$

9. Consider the region contained under the graph of $f(x) = x+1$ between $x = 1$ and $x = 2$. Since the graph of f is a straight line, you can compute the area of this region by dividing it into a rectangle and a triangle. Do so. Then arrive at the same answer by using Fact 40.3.

This region can be divided into a 1×2 rectangle and a triangle with base 1 and height 1, as shown on the left. Thus its area is $1\cdot 2 + \frac{1}{2}\cdot 1\cdot 1 = 5/2$ square units.



In the area formula, $\Delta x = (2-1)/n = 1/n$ and $x_k = 1 + k\Delta x = 1 + k/n$. Then:

$$\begin{aligned} A &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(1+\frac{k}{n}\right)\frac{1}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{k}{n}+1\right)\frac{1}{n} \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{n}+\frac{k}{n^2}\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{2}{n}+\sum_{k=1}^{n}\frac{k}{n^2}\right) = \lim_{n\to\infty}\left(n\frac{2}{n}+\frac{1}{n^2}\sum_{k=1}^{n} k\right) \\ &= \lim_{n\to\infty}\left(2+\frac{1}{n^2}\frac{n(n-1)}{2}\right) = \lim_{n\to\infty}\left(2+\frac{1}{2}\frac{n^2-n}{n^2}\right) = 2+\frac{1}{2}\cdot 1 = \frac{5}{2}. \end{aligned}$$