CHAPTER 6

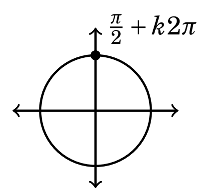
Inverse Trigonometric Functions

In solving trigonometric equations (as in Section 3.4) we typically reduce a complex trigonometric equation to a simple one, such as $\sin(\theta) = 1$, with the variable θ occurring inside the trig function. Then we have to reason backwards and ask ourselves "*What value of* θ *makes* $\sin(\theta) = 1$*?*" Reflecting on the unit circle, we might arrive at the solution $\theta = \frac{\pi}{2}$.

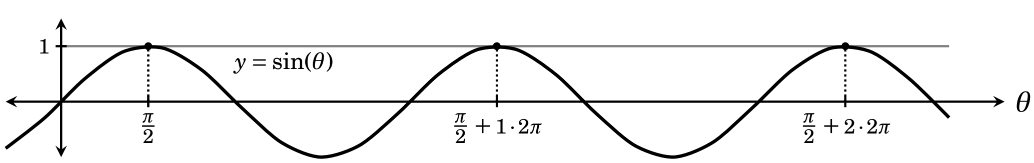
This is exactly the kind of backwards thinking needed to mentally evaluate inverse functions. We are in essence treating sin as if it had an inverse and reasoning as

$$\sin^{-1}(1) = \left(\begin{array}{l} \text{angle } \theta \text{ for} \\ \text{which } \sin(\theta) = 1 \end{array} \right) = \frac{\pi}{2}.$$

But there is a slight problem with this. There are *many* values θ for which $\sin(\theta) = 1$, namely $\theta = \frac{\pi}{2} + k2\pi$ for any integer k. This is clear from the unit circle. Start at $\theta = \frac{\pi}{2}$ radians and take k laps (of length 2π) around the circle to return to the same point, but at $\theta = \frac{\pi}{2} + k2\pi$ radians. We still have $\sin\left(\frac{\pi}{2} + k2\pi\right) = 1$.



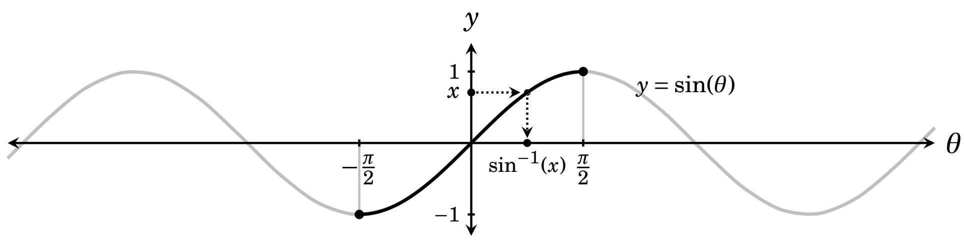
This is also clear from the graph of sin, as $\sin(\theta) = 1$ at each $\theta = \frac{\pi}{2} + k2\pi$. The graph reveals the crux of our problem: sin is not one-to-one because the horizontal line $y = 1$ meets it at every $\theta = \frac{\pi}{2} + k2\pi$. So sin has no inverse.



We have a dilemma. An inverse $\sin^{-1}$ would be useful, but it doesn't exist. We will overcome this by restricting the domain of sin.

6.1 The Function $\sin^{-1}$

The function sin is not one-to-one, so technically it has no inverse. But in the diagram below, the part of its graph (bold) over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *is* a one-to-one function. *This* function has an inverse; we will call it $\sin^{-1}$. For input, $\sin^{-1}$ accepts values of x between -1 and 1 (the *outputs* of sin). Given this input, the output of $\sin^{-1}(x)$ is the angle θ for which $\sin(\theta) = x$, with the additional stipulation $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. (See the diagram below.)



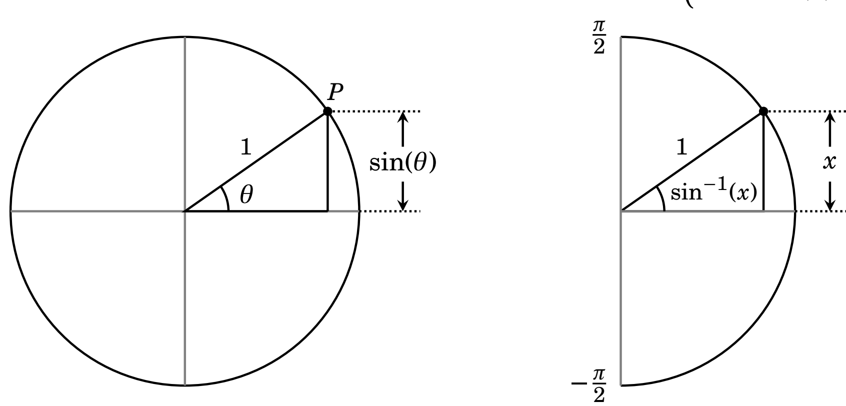
To repeat, $\sin^{-1}$ is the inverse of sin with its domain restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Notice $\sin^{-1}$ has domain $[-1, 1]$, and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

There is a simple way to visualize $\sin^{-1}$. The left side of the box below reminds us that $\sin(\theta)$ is the height of the point P at θ on the unit circle: For input θ we get an output $\sin(\theta)$, the y-coordinate of P. The right side of the box reverses this. The input $-1 \le x \le 1$ is the height of a point on the unit circle; the output $\sin^{-1}(x)$ is the angle $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ for which $\sin(\theta) = x$.

The functions sin and $\sin^{-1}$

$\sin(\theta) = \big(\ y\text{-coordinate of point } P\ \big)$

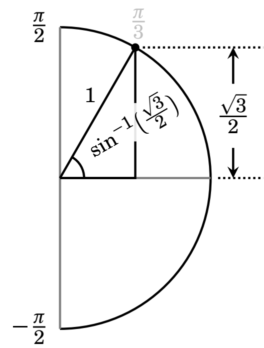
$$\sin^{-1}(x) = \left(\begin{array}{l} \text{angle } \theta \\ \text{with } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{and } \sin(\theta) = x \end{array}\right)$$



This is a *picture* of $\sin^{-1}$. It says that a right triangle with hypotenuse of length 1 and opposite leg of length x has an angle measuring $\sin^{-1}(x)$ radians. From this we can mentally work out $\sin^{-1}(x)$ for many x.

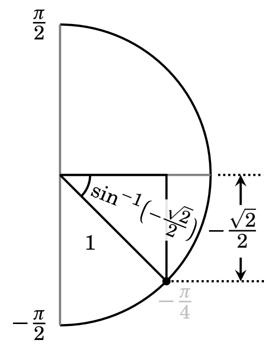
Here are a few examples. We will start out drawing a picture for each one, though you will quickly reach the point of bypassing the picture in favor of visualizing the situation in the mind's eye.

First let's find $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. Draw a half unit circle representing angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and put a point on it at height $\frac{\sqrt{3}}{2}$. This forms a familiar 30-60-90 triangle with the 60° (or $\frac{\pi}{3}$) angle at the origin. Thus



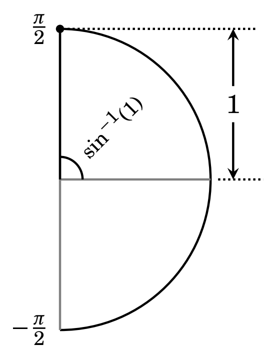
$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \left(\begin{array}{l}\text{angle } \theta \text{ for} \\ \text{which } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{and } \sin(\theta) = \frac{\sqrt{3}}{2}\end{array}\right) = \frac{\pi}{3}.$$

For another example, let's find $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. First we draw the half-circle and locate the point at height $-\frac{\sqrt{2}}{2}$, this time below the x-axis. This forms an angle of radian measure $-\frac{\pi}{4}$. Thus



$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \left(\begin{array}{l}\text{angle } \theta \text{ for} \\ \text{which } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{and } \sin(\theta) = -\frac{\sqrt{2}}{2}\end{array}\right) = -\frac{\pi}{4}.$$

Now let's do $\sin^{-1}(1)$. The point on the half-circle at height 1 forms the angle $\frac{\pi}{2}$, so



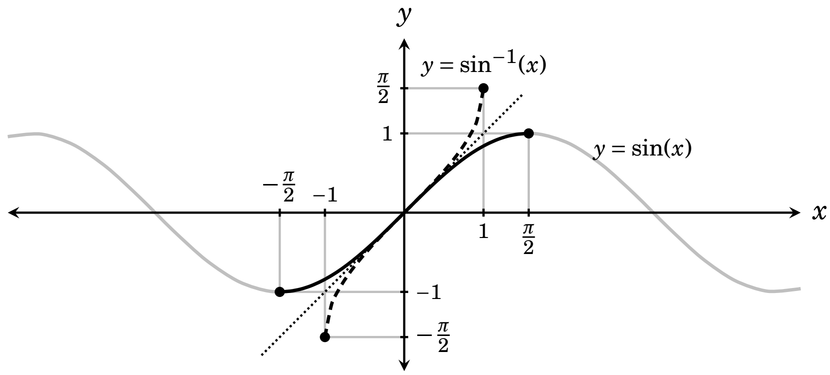
$$\sin^{-1}(1) = \left(\begin{array}{l}\text{angle } \theta \text{ for} \\ \text{which } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{and } \sin(\theta) = 1\end{array}\right) = \frac{\pi}{2}.$$

From the same picture we can see $\sin^{-1}(-1) = -\frac{\pi}{2}$ and $\sin^{-1}(0) = 0$.

Of course we can't do every such problem mentally. Consider $\sin^{-1}\left(\frac{1}{3}\right)$. This poses a problem because we can't think of a θ for which $\sin(\theta) = \frac{1}{3}$. But using the $\boxed{\sin^{-1}}$ button on your calculator gives $\sin^{-1}\left(\frac{1}{3}\right) \approx 0.39836909$.[1] Use your calculator sparingly. Do not allow it to obstruct the simple meaning of $\sin^{-1}$.

[1] Your calculator should allow you to toggle between *radian* and *degree* mode. Test this by working out $\sin^{-1}(1)$. With degree mode you should get $\sin^{-1}(1) = 90°$. Radian mode gives $\sin^{-1}(1) = 1.5707963$ (which is $\frac{\pi}{2}$).

Finally, let's investigate the graph of $\sin^{-1}$. Below is the graph of $y = \sin(x)$ with the part on domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ drawn bold. Reflecting this across the line $x = y$ gives the graph of $y = \sin^{-1}(x)$, drawn dashed.



Erasing the clutter, the graph of $\sin^{-1}(x)$ is shown again as Figure 6.1. Notice that the domain of $\sin^{-1}$ is the interval $[-1, 1]$. The range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

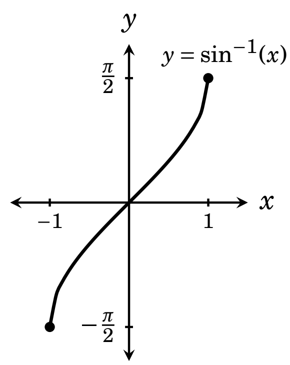


Figure 6.1. The graph of $\sin^{-1}(x)$. It has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

We remark that $\sin\left(\sin^{-1}(x)\right) = x$ for any real number x in the domain of $\sin^{-1}$, in accordance with the property $f\left(f^{-1}(x)\right) = x$. But we caution that $\sin^{-1}\left(\sin(x)\right) = x$ is not always true. Indeed consider $x = 2\pi$. Then $\sin^{-1}\left(\sin(2\pi)\right) = \sin^{-1}(0) = 0 \neq 2\pi$, so $\sin^{-1}(\sin(x)) \neq x$ in this particular case. The reason for this apparent contradiction is that sin does not really have an inverse. The function $\sin^{-1}$ is the inverse of sin *with a restricted domain*. The value $x = 2\pi$ is not in the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of sin and thus lies outside the range of $\sin^{-1}$.

Example 6.1 Find all solutions of the equation $4\sin^2(x)+3\sin(x)=1$ that are in the interval $[0,2\pi]$.

Because of the second power of sin, it seems reasonable to try to solve by factoring. For this we need zero on one side, so we rewrite as

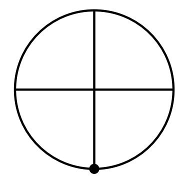
$$4\sin^2(x)+3\sin(x)-1=0.$$

This factors as

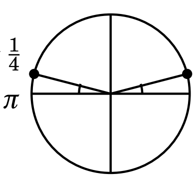
$$\big(4\sin(x)-1\big)\big(\sin(x)+1\big)=0.$$

So x must be such that it makes one or the other factor zero. We consider these cases separately.

First, if $\sin(x)+1=0$, then $\sin(x)=-1$. There is only one x in $[0,2\pi]$ that makes this true, and that is $x=\frac{3\pi}{2}$ which is immediately evident from the unit circle. Therefore one solution to the equation is $x=\frac{3\pi}{2}$.



Next suppose $4\sin(x)-1=0$, so $\sin(x)=\frac{1}{4}$. We know no angle x making $\sin(x)=\frac{1}{4}$, so we cannot solve this from direct inspection of the unit circle. But we can take $\sin^{-1}$ of both sides of $\sin(x)=\frac{1}{4}$, and doing this yields a solution $x=\sin^{-1}\left(\frac{1}{4}\right)$, which is the indicated point on the circle at height $\frac{1}{4}$, in the first quadrant. There is another point at height $\frac{1}{4}$, in the second quadrant, and this is at $\pi-\sin^{-1}\left(\frac{1}{4}\right)$. Taking sin of either of these angles results in $\frac{1}{4}$ so they are the solutions of $\sin(x)=\frac{1}{4}$.



So the equation has three solutions, $x=\frac{3\pi}{2}$, $\sin^{-1}\left(\frac{1}{4}\right)$, and $\pi-\sin^{-1}\left(\frac{1}{4}\right)$. Resorting to a calculator, $x\approx 4.7123889,\ 0.25268024,\ 2.88891239$. ✍

Exercises for Section 6.1

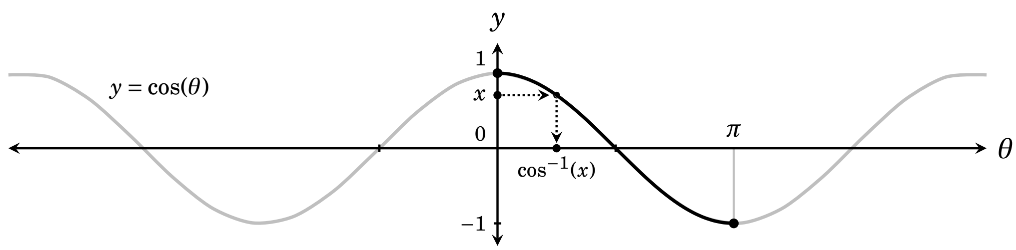
Work these problems mentally, without a calculator.

1. $\sin^{-1}(-1)$ **2.** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **3.** $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ **4.** $\sin^{-1}\left(-\frac{1}{2}\right)$

5. $\sin^{-1}(1)$ **6.** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **7.** $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ **8.** $\sin^{-1}\left(\frac{1}{2}\right)$

6.2 The Function $\cos^{-1}$

The cos function is certainly not one-to-one, so it doesn't have an inverse – unless we restrict its domain. The diagram below reveals that $\cos$ is one-to-one on the domain $[0,\pi]$. (This part of the graph is drawn bold.)



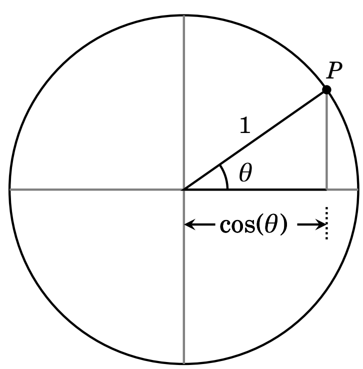
We define $\cos^{-1}$ to be the inverse of the function $\cos$ with domain $[0,\pi]$.

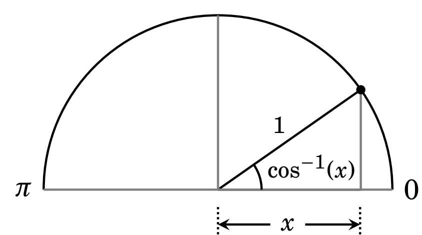
The boxed diagrams below are visual descriptions of $\cos$ and $\cos^{-1}$. For an angle θ we know that $\cos(\theta)$ is the x-coordinate of the point P at θ on the unit circle. Reversing this, to find $\cos^{-1}(x)$, envision x as the x-coordinate of a point on the upper half circle (encompassing all angles $0 \le \theta \le \pi$). Then $\cos^{-1}(x)$ is the angle θ between 0 and π for which $\cos(\theta) = x$.

The functions cos and $\cos^{-1}$

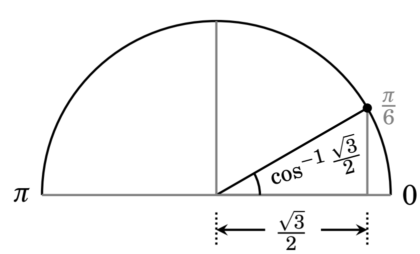
$\cos(\theta) = \big(\ x\text{-coordinate of point } P\ \big)$

$$\cos^{-1}(x) = \left(\begin{array}{l} \text{angle } \theta \text{ for} \\ \text{which } 0 \le \theta \le \pi \\ \text{and } \cos(\theta) = x \end{array} \right)$$

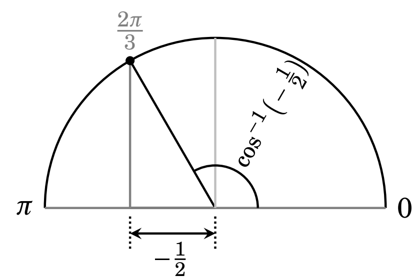




To illustrate, let's compute $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. We first draw a point on the upper circle with x-coordinate $\frac{\sqrt{3}}{2}$. Our experience with the unit circle tells us that this forms an angle of $\frac{\pi}{6}$ radians. Thus $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$.

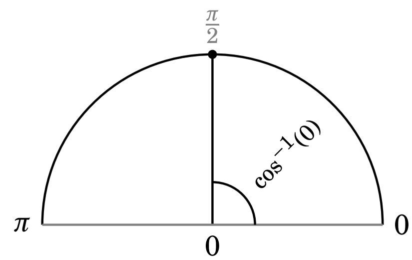


Next consider the problem of $\cos^{-1}\left(-\frac{1}{2}\right)$. We first locate the point on the upper circle with x-coordinate $-\frac{1}{2}$. Knowledge of the unit circle tells us that this forms an angle of $\frac{2\pi}{3}$ radians, so $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

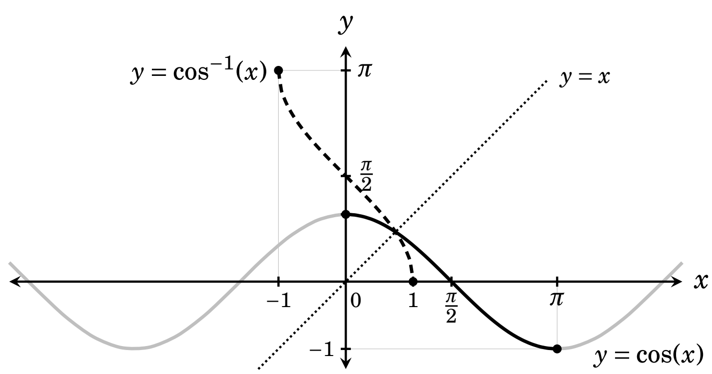


Now let's do $\cos^{-1}(0)$. The point on the upper circle with x-coordinate 0 is at $\frac{\pi}{2}$ radians. Therefore $\cos^{-1}(0) = \frac{\pi}{2}$.

You can also see from this drawing how $\cos^{-1}(1) = 0$ and $\cos^{-1}(-1) = \pi$.



Now that we can compute $\cos^{-1}(x)$ let's think about its graph. Below we get the graph of $y = \cos^{-1}(x)$ (dashed) by reflecting across the line $y = x$ the relevant part of the graph of $y = \cos(x)$ (bold).



Erasing the preliminary steps, we get our final graph, in Figure 6.2.

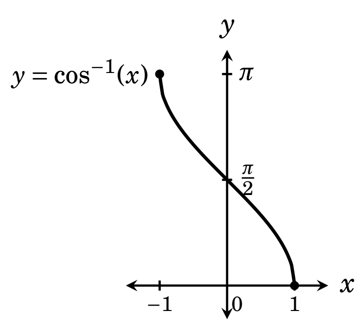
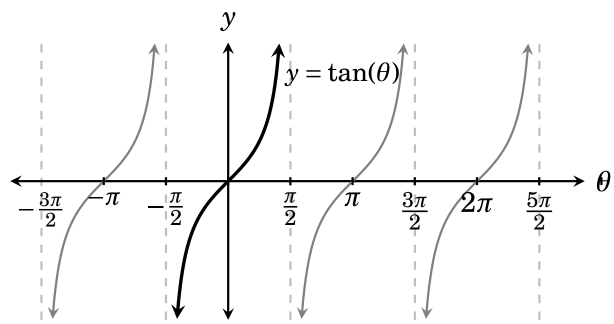


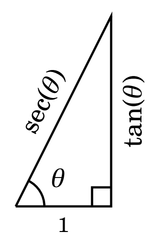
Figure 6.2. The graph of $\cos^{-1}(x)$. It has domain is $[-1,1]$ and range $[0,\pi]$.

6.3 The Functions $\tan^{-1}$ and $\sec^{-1}$

Now we will develop the functions $\tan^{-1}$ and $\sec^{-1}$, beginning with $\tan^{-1}$. The function tan is definitely not one-to-one, but its graph suggests that we can make it one-to-one by restricting its domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Our visual descriptions of $\tan^{-1}$ and $\sec^{-1}$ will use a right triangle whose adjacent side has length 1, shown here. The opposite side is $\text{OPP} = \frac{\text{OPP}}{1} = \frac{\text{OPP}}{\text{ADJ}} = \tan(\theta)$, while the hypotenuse is $\text{HYP} = \frac{\text{HYP}}{1} = \frac{\text{HYP}}{\text{ADJ}} = \sec(\theta)$. This is a picture of $\tan(\theta)$ and $\sec(\theta)$. The number $\tan(\theta)$ is the length of the opposite side of the triangle; $\sec(\theta)$ is its hypotenuse.

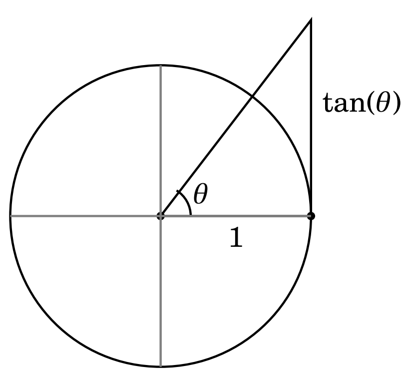


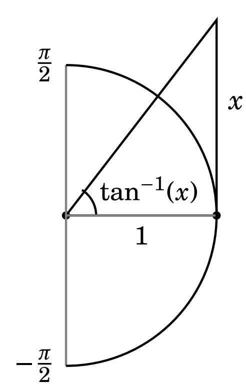
Placing this on the unit circle gives a visual description of both $\tan(\theta)$ and $\tan^{-1}(x)$. The left side of the box (below) states that $\tan(\theta)$ is the opposite side of our triangle. The right side reverses this: If the opposite side is x, then the angle is $\tan^{-1}(x)$.

The functions tan and $\tan^{-1}$

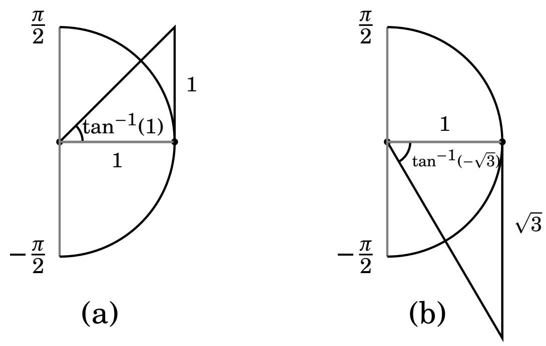
$\tan(\theta) = $ (opposite side of triangle)

$$\tan^{-1}(x) = \left(\begin{array}{l} \text{angle } \theta \text{ with} \\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \text{and } \tan(\theta) = x \end{array} \right)$$



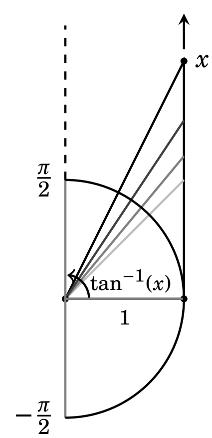


As an example, consider $\tan^{-1}(1)$. You can probably already do this mentally without resorting to a picture: $\tan^{-1}(1)$ is the angle θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\tan(\theta) = 1$. As $\tan\left(\frac{\pi}{4}\right) = 1$, we get $\tan^{-1}(1) = \frac{\pi}{4}$. Drawing (a) below reinforces this. It shows a triangle with adjacent side 1 and opposite side $x = 1$. This is a 45-45-90 triangle, so the angle $\tan^{-1}(1)$ must be $\frac{\pi}{4}$.



Now think about $\tan^{-1}\left(-\sqrt{3}\right)$. Drawing (b) shows the relevant triangle. Because $-\sqrt{3}$ is negative, we orient the triangle so that the vertical side is below the x-axis. This is a 30-60-90 triangle with the angle $\tan^{-1}\left(-\sqrt{3}\right)$ at the 60° corner. Therefore $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$.

Notice that as the opposite side x of our triangle grows larger, the hypotenuse becomes more vertical, and the angle $\tan^{-1}(x)$ rotates closer to 90°, or $\frac{\pi}{2}$ radians. Thus as x increases to ∞ the value of $\tan^{-1}(x)$ increases, approaching $\frac{\pi}{2}$. This is reflected in the graph of $\tan^{-1}$ below – the line $y = \frac{\pi}{2}$ is a horizontal asymptote. (This graph is the reflection across the line $y = x$ of the graph of $y = \tan(x)$ with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The vertical asymptote $x = \frac{\pi}{2}$ of $\tan(x)$ reflects to the horizontal asymptote $y = \frac{\pi}{2}$.) If x approaches $-\infty$, the triangle is below the x-axis, and the angle $\tan^{-1}(x)$ approaches $-\frac{\pi}{2}$.



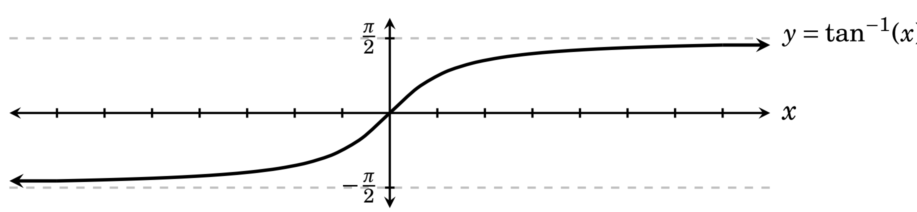
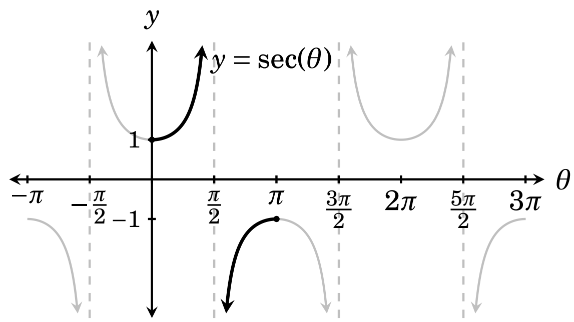


Figure 6.3. The graph of $\tan^{-1}(x)$. Lines $y = \pm\frac{\pi}{2}$ are horizontal asymptotes.

Now we explore the function $\sec^{-1}(x)$. As usual, because $\sec(x)$ is not one-to-one we will have to restrict its domain in order to get an inverse. Its graph suggests that we should restrict it to the interval $[0,\pi]$. Actually, to be absolutely precise, the point $x=\frac{\pi}{2}$ is not in the domain, so we restrict sec to $[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$. This is indicated by the bold part of the graph below.

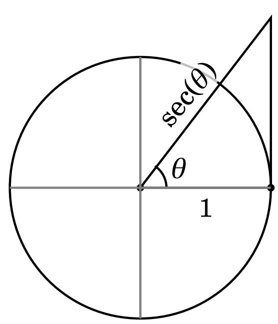


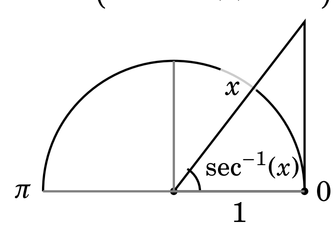
We therefore define $\sec^{-1}(x)$ to be the inverse of the function $\sec(x)$ with restricted domain $[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi]$. At the beginning of this section we remarked that $\sec(\theta)$ is the length of the hypotenuse of a right triangle with angle θ and adjacent side having length 1. This gives the following interpretation, at least for values of θ with $0\le\theta<\frac{\pi}{2}$.

The functions sec and $\sec^{-1}$

$\sec(\theta) = $ (hypotenuse of triangle)

$$\sec^{-1}(x) = \left(\begin{array}{l} \text{angle } \theta \text{ for} \\ \text{which } 0\le\theta\le\pi \\ \text{and } \sec(\theta)=x \end{array} \right)$$

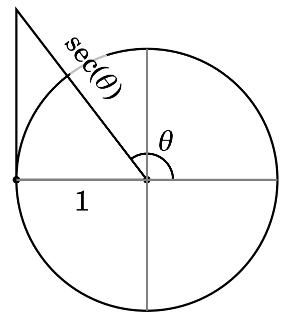




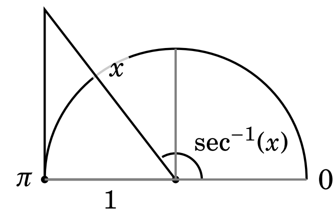
In other words, if the above triangle has hypotenuse x, then $\sec^{-1}(x)$ is the radian measure its angle.

One slightly unsettling thing about this situation is that the picture changes when $\frac{\pi}{2}<\theta\le\pi$, when θ is in the second quadrant. Then the triangle flips to the left of the y-axis, as illustrated below. Note $\sec(\theta)$ is negative for these θ, so we have to interpret the hypotenuse as a negative number. Thus the diagram on the right shows the angle $\sec^{-1}(x)$ for *negative values of x.*

$\sec(\theta) = -(\text{hypotenuse of triangle})$



$$\sec^{-1}(x) = \left(\begin{array}{l}\text{angle } \theta \text{ for}\\ \text{which } 0 \le \theta \le \pi\\ \text{and } \sec(\theta) = x\end{array}\right)$$



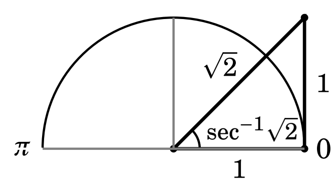
(Here x is regarded as negative, so this is the picture of $\sec^{-1}(x)$ for *negative* x.)

Regardless of the diagrams, $\sec^{-1}(x)$ is always the angle θ between 0 and π with $\sec(\theta) = x$. Thus $\sec^{-1}(x) \ge 0$, though its input can be negative.

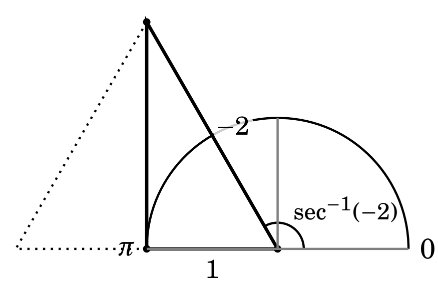
Example 6.2 Find $\sec^{-1}\left(\sqrt{2}\right)$, $\sec^{-1}(-2)$ and $\sec^{-1}(4)$.

Each of these can be solved without resorting to a diagram. Below we work them first without a diagram, but then add the diagram to highlight the geometric meaning of the inverse secant function.

First we find $\sec^{-1}\left(\sqrt{2}\right)$. If $\theta = \sec^{-1}\left(\sqrt{2}\right)$, then $0 \le \theta \le \pi$ and $\sec(\theta) = \sqrt{2}$, which means $\cos(\theta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. From this we see $\theta = \frac{\pi}{4}$, that is, $\sec^{-1}\left(\sqrt{2}\right) = \frac{\pi}{4}$. The diagram reinforces this. The triangle with hypotenuse $\sqrt{2}$ is a 45-45-90 triangle, with angle $\frac{\pi}{4}$.

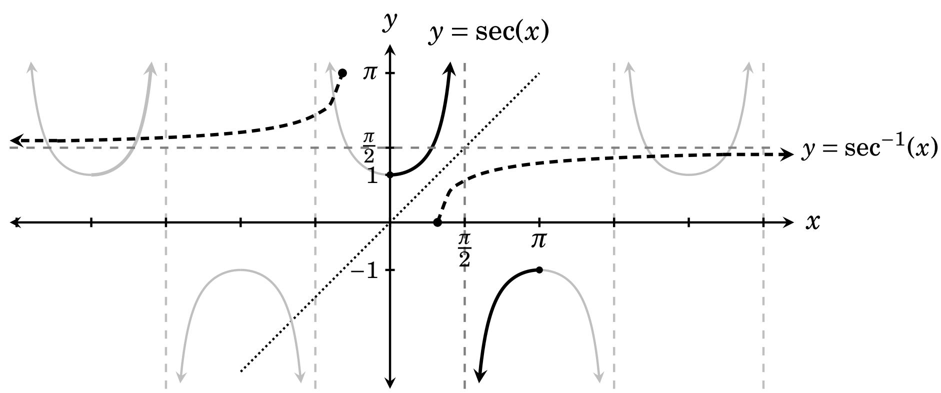


Now for $\sec^{-1}(-2)$. If $\theta = \sec^{-1}(-2)$, then $0 \le \theta \le \pi$ and $\sec(\theta) = -2$, or $\cos(\theta) = -\frac{1}{2}$. From this see $\theta = \frac{2\pi}{3}$, so $\sec^{-1}(-2) = \frac{2\pi}{3}$. Again a diagram reinforces this. We interpret the input of -2 as the hypotenuse of a triangle in the second quadrant. Here it happens that the triangle is half of an equiangular triangle with sides of length 2. We read off $\sec^{-1}(-2) = 120°$, or $\frac{2\pi}{3}$ radians.



Finally, consider $\sec^{-1}(4)$. If $\theta = \sec^{-1}(4)$, then $\sec(\theta) = 4$ and $\cos(\theta) = \frac{1}{4}$. Here there is no familiar angle θ for which $\cos(\theta) = \frac{1}{4}$. We have to resort to a calculator approximation $\sec^{-1}(4) \approx 1.31811607$. (This is in *radians*. In degree mode your calculator will give $\sec^{-1}(4) \approx 75.52248781$ degrees.) ✍

Now we investigate the graph of $\sec^{-1}(x)$. Our strategy is to use the fact that the graph of the inverse of a function is the function's graph reflected across the line $y = x$. In this case we start with the graph of $y = \sec(x)$ restricted to the domain $[0,\pi]$, shown bold below. (The other portions of $y = \sec(x)$ are drawn in light gray.) This bold graph $y = \sec(x)$ reflects across the line $y = x$ (dotted) to the dashed graph, which is the graph of $y = \sec^{-1}(x)$. Notice how the vertical asymptote $x = \frac{\pi}{2}$ of $\sec(x)$ reflects to a horizontal asymptote $y = \frac{\pi}{2}$ of $\sec^{-1}(x)$.



Erasing all the junk, we get the clean graph below. Notice that the domain of $\sec^{-1}(x)$ is $(-\infty,-1]\cup[1,\infty)$. The range is $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$.

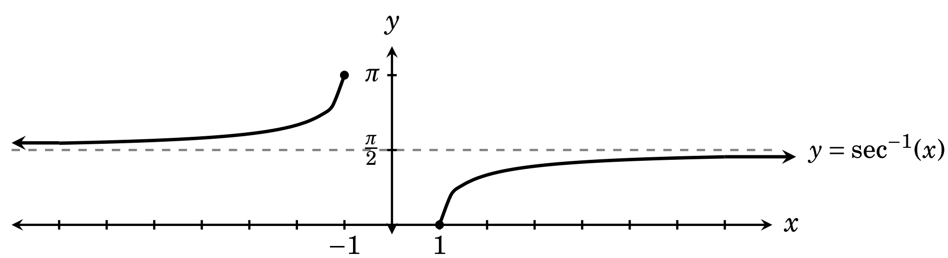


Figure 6.4. The graph of $\sec^{-1}(x)$. The line $y = \frac{\pi}{2}$ is a horizontal asymptote.

Take note that the interval $(-1,1)$ is *not* a part of the domain of $\sec^{-1}(x)$. This is because all outputs of $\sec(x)$ are either greater than or equal to 1, or less than or equal to 1. As the inputs of $\sec^{-1}(x)$ are the outputs of $\sec(x)$, no numbers in $(-1,1)$ can be plugged into $\sec^{-1}(x)$.

This follows also from our diagrams. The x in $\sec^{-1}(x)$ is identified with the hypotenuse of a right triangle on the unit circle, which never has a length between 1 and -1.

Exercises for Section 6.3

These problems are for both Sections 6.3 and 6.2.

Evaluate the following inverse trig functions. It is important to do them without a calculator – this will greatly sharpen your understanding of inverse trig functions.

1. $\cos^{-1}\left(\frac{1}{2}\right)$
2. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
3. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
4. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

5. $\tan^{-1}(0)$
6. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
7. $\tan^{-1}\left(\sqrt{3}\right)$
8. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

9. $\tan^{-1}(-1)$
10. $\sec^{-1}(1)$
11. $\sec^{-1}(2)$
12. $\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

13. $\sec^{-1}\left(-\sqrt{2}\right)$
14. $\sec^{-1}(-1)$
15. $\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right)$
16. $\sec^{-1}\left(\frac{2}{\sqrt{2}}\right)$

17. $\sin^{-1}\left(\sin(5\pi)\right)$
18. $\sin^{-1}\left(\sin\left(\frac{3\pi}{2}\right)\right)$

19. $\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right)$
20. $\tan^{-1}\left(\tan\left(\frac{4\pi}{3}\right)\right)$

21. $\sec^{-1}\left(\sec\left(\frac{5\pi}{4}\right)\right)$
22. $\sec^{-1}\left(\sec\left(\frac{4\pi}{3}\right)\right)$

23. $\cos^{-1}\left(\cos(8\pi)\right)$
24. $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right)$

25. Solve the equation $\tan^2(x) = \frac{1}{3}$. (Use a calculator to approximate the final answer, if you wish.)

26. Find all solutions of $\tan^2(x) + \tan(x) - 2 = 0$ that are in the interval $[-\pi, \pi]$. (Use a calculator to approximate the final answer, if you wish.)

27. Find all solutions of $5x\sin(x) = x$ that are in the interval $[-\pi, \pi]$. (Use a calculator to approximate the final answer, if you wish.)

28. Find all solutions of the equation $3\cos(x)\sin(x) + 6\cos(x) - \sin(x) - 2 = 0$ that are in the interval $[0, 2\pi]$. (Use a calculator to approximate the final answer, if you wish.)

29. Explain why $\cos^{-1}(x) = \frac{\pi}{2} - \sin^{-1}(x)$ for all x in the domain of $\cos^{-1}$ and $\sin^{-1}$.

30. Explain why $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$ for all $x \geq 0$.

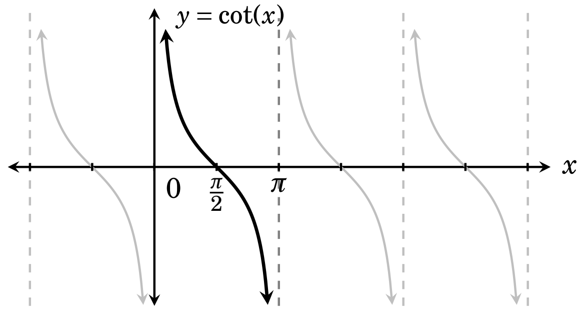
31. Explain why $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$.

32. Explain why $\cos^{-1}(x) + \cos^{-1}(-x) = \pi$ for all x in the domain of $\cos^{-1}$.

33. Explain why $\sin^{-1}(x) + \sin^{-1}(-x) = 0$ for all x in the domain of $\sin^{-1}$.

6.4 The Functions $\cot^{-1}$ and $\csc^{-1}$

These functions are occasionally useful. But we will see that they are expressible in terms of $\tan^{-1}$ and $\sec^{-1}$, so they need only a brief mention.



First, consider $\cot^{-1}(x)$. The above graph of $\cot(x)$ shows that it is one-to-one on the interval $[0,\pi]$. We thus define $\cot^{-1}(x)$ as the inverse of $\cot(x)$ with its domain restricted to $[0,\pi]$. Below is the graph of $y = \cot^{-1}(x)$, which is the reflection of the bold part of $y = \cot(x)$ (above) across the line $y = x$. Notice how the vertical asymptote $y = \pi$ of $\cot(x)$ reflects across to a *horizontal* asymptote of $\cot^{-1}(x)$. Likewise the y-axis (a vertical asymptote of $\cot(x)$) reflects across to the x-axis, a horizontal asymptote of $\cot^{-1}(x)$.

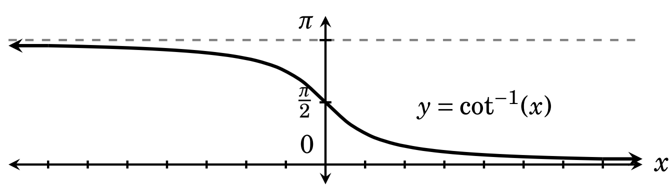


Figure 6.5. The function $\cot^{-1}(x)$ has domain $\mathbb{R}$ and the range $[0,\pi]$.

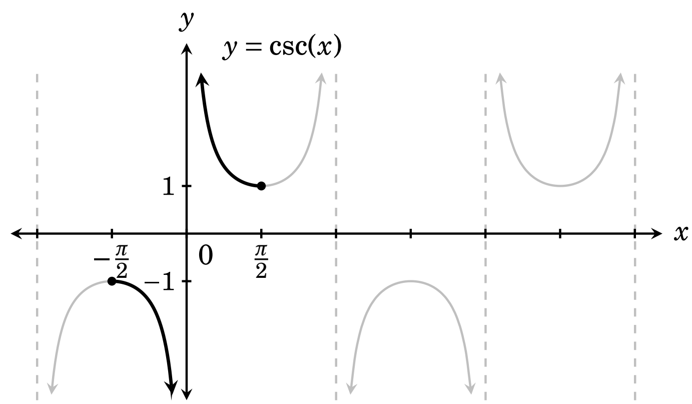
Given an input x, the corresponding output $\cot^{-1}(x)$ is the angle θ for which $0 < \theta < \pi$ and $\cot(\theta) = x$.

Compare the graphs of $\cot^{-1}(x)$ in Figure 6.5 and $\tan^{-1}(x)$ in Figure 6.3 on page 106. It appears that the graph of $\cot^{-1}(x)$ is that of $\tan^{-1}(x)$ reflected across the x-axis and then moved up $\frac{\pi}{2}$ units. That is, we might guess

$$\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x).$$

This is indeed the case, as you are invited to verify. The fact that $\cot^{-1}(x)$ can be expressed in terms of $\tan^{-1}(x)$ is one reason that we do not need to invest much mental energy into the intricacies of $\cot^{-1}(x)$, as least as long as we have a good grip on $\tan^{-1}(x)$.

Now we'll investigate the last remaining inverse trig function, $\csc^{-1}(x)$. We start with the graph of $\csc(x)$, below. This function is one-to-one on the domain $\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$, where the point $x=0$ is not included because it is not in the domain of $\csc(x)$. This part of the graph is shown bold.



The function $\csc^{-1}(x)$ is defined to be the inverse of $\csc(x)$ on this restricted domain. Reflecting the bold part of the graph of $\csc(x)$ across the line $y=x$ gives the graph of $\csc^{-1}(x)$, below.

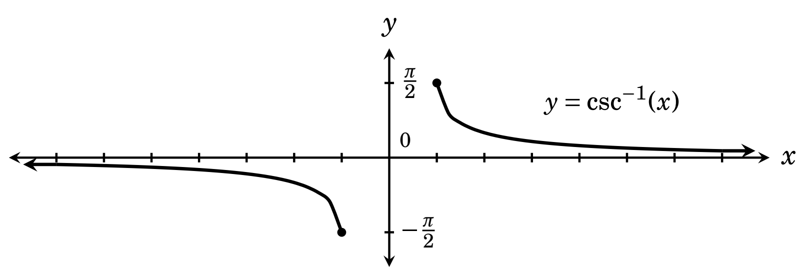


Figure 6.6. The graph of $\csc^{-1}(x)$. Its domain is $(-\infty,-1]\cup[1,\infty)$ and its range is $\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$. The x-axis is a horizontal asymptote.

Given an input x, the corresponding output $\csc^{-1}(x)$ is the angle θ for which $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ and $\csc(\theta)=x$.

Comparing the graphs of $\csc^{-1}(x)$ in Figure 6.6 and $\sec^{-1}(x)$ in Figure 6.4 on page 109, it looks as if the graph of $\csc^{-1}(x)$ is the graph of $\sec^{-1}(x)$ reflected across the x-axis, then moved up $\frac{\pi}{2}$ units. That is, we might guess

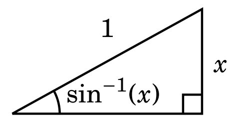
$$\csc^{-1}(x)=\frac{\pi}{2}-\sec^{-1}(x).$$

This is indeed the case, as you can verify. Because $\csc^{-1}(x)$ can be expressed in terms of $\sec^{-1}(x)$, we can downplay the importance of $\csc^{-1}(x)$.

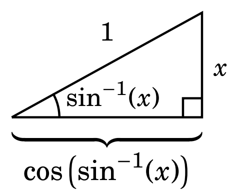
6.5 Simplifications

On page 101 we noted that $\sin\left(\sin^{-1}(x)\right) = x$. Obviously the right side of this equation is simpler than the left side. Taking $\sin$ of $\sin^{-1}(x)$ wipes out the trig functions.

What if we did $\cos\left(\sin^{-1}(x)\right)$? This too simplifies dramatically, but the answer is not x. To understand how this expression simplifies, let's examine the angle $\sin^{-1}(x)$ that we are taking $\cos$ of. We saw in Section 6.1 how to draw a picture of $\sin^{-1}(x)$. It is the angle of the following triangle whose hypotenuse has length 1 and whose opposite side has length x.



Taking $\cos$ of this angle, we get the length of the adjacent side, as follows.



Now, we can solve for this adjacent side using the Pythagorean theorem:

$$\begin{aligned}\left(\cos\left(\sin^{-1}(x)\right)\right)^2 + x^2 &= 1^2\\ \left(\cos\left(\sin^{-1}(x)\right)\right)^2 &= 1 - x^2\\ \cos\left(\sin^{-1}(x)\right) &= \sqrt{1-x^2}.\end{aligned}$$

Therefore we obtain the simplification

$$\cos\left(\sin^{-1}(x)\right) = \sqrt{1-x^2},$$

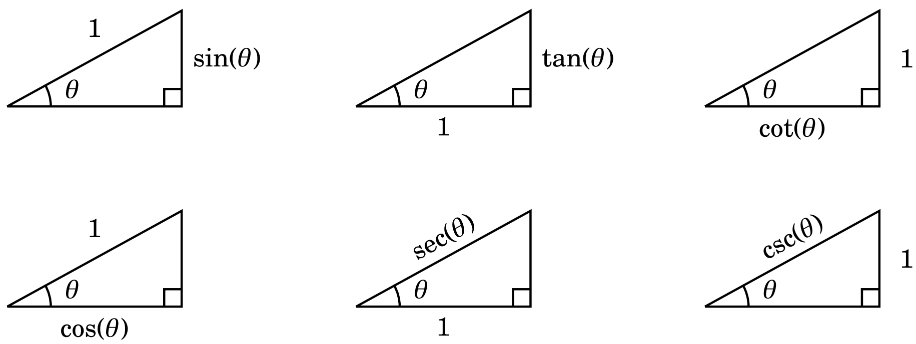
which holds true for all x in the domain $[-1,1]$ of $\sin^{-1}(x)$. In essence, taking $\cos$ of the angle $\sin^{-1}(x)$ wipes out the trig functions, leaving a simpler algebraic expression.

There will be occasions where such simplifications are very useful.

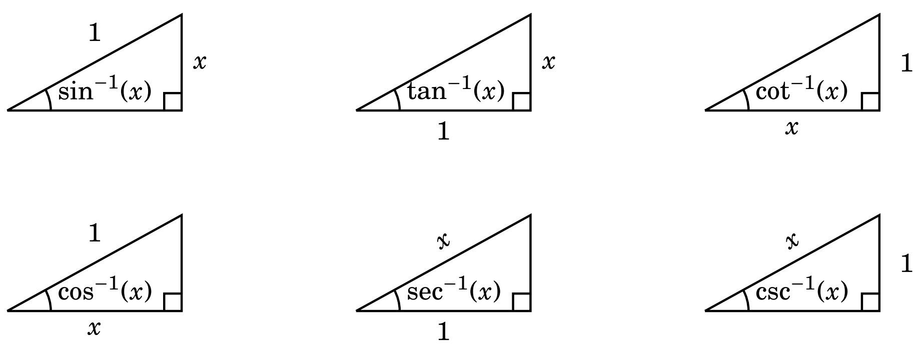
This section explains how to simplify expressions such as $\cos\left(\sin^{-1}(x)\right)$, or $\cos\left(\tan^{-1}(x)\right)$, or $\sec\left(\tan^{-1}(x)\right)$, etc., that involve the composition of a trig function with an inverse trig function. In every case the answer can be found by applying the Pythagorean theorem to an appropriate triangle.

Success at doing this revolves around understanding the geometric, triangle interpretation of the six trig functions. This is summarized in the six triangles below. In each case a trig function is interpreted as the length of a triangle edge, where one other edge has length 1.

For example, in the first triangle, OPP $= \frac{\text{OPP}}{1} = \frac{\text{OPP}}{\text{ADJ}} = \sin(\theta)$, that is, the opposite side has length $\sin(\theta)$. You should check that the remaining triangles are labeled correctly.



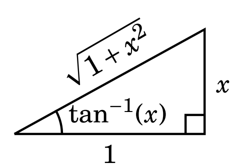
Reversing input and output (as we have learned to do in this chapter) yields corresponding triangles for each of the six inverse trig functions.



Now we will use these to solve a few instances of simplifications.

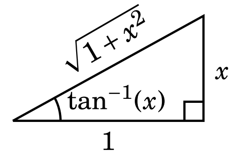
Example 6.3 Simplify $\sec\left(\tan^{-1}(x)\right)$.

Start with the triangle for $\tan^{-1}(x)$, redrawn here. By the Pythagorean theorem, the hypotenuse is $\sqrt{1+x^2}$, as labeled. Now, $\sec\left(\tan^{-1}(x)\right) = \frac{\text{HYP}}{\text{ADJ}} = \frac{\text{HYP}}{1} = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}$.



Therefore we have obtained $\sec\left(\tan^{-1}(x)\right) = \sqrt{1+x^2}$. ✍

Example 6.4 Simplify $\sin\left(\tan^{-1}(x)\right)$.

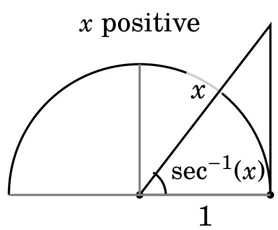


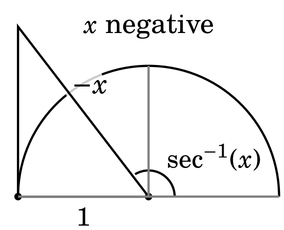
Again we start with the triangle for $\tan^{-1}(x)$. As before, the Pythagorean theorem says the hypotenuse has length $\sqrt{1+x^2}$. Now, $\sin\left(\tan^{-1}(x)\right) = \frac{\text{OPP}}{\text{HYP}} = \dfrac{x}{\sqrt{1+x^2}} = \dfrac{x}{\sqrt{1+x^2}}$.

Therefore we have obtained our answer $\sin\left(\tan^{-1}(x)\right) = \dfrac{x}{\sqrt{1+x^2}}$. ✍

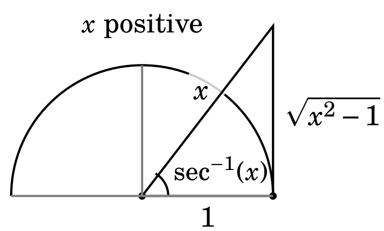
Example 6.5 Simplify $\tan\left(\sec^{-1}(x)\right)$.

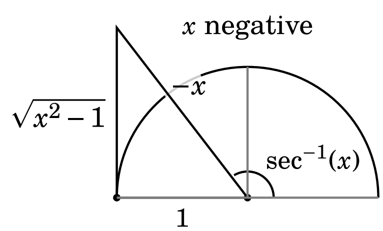
We start by drawing the triangle for $\sec^{-1}(x)$. This time things are slightly trickier. Recall from Section 6.3 that we interpret the hypotenuse length x to be positive or negative depending on whether the triangle is in the first or second quadrant.





Either way, the Pythagorean theorem says the opposite side has length $\sqrt{x^2-1}$, which is positive, regardless of the sign of x.





For the triangle in the first quadrant, $\tan\left(\sec^{-1}(x)\right) = \frac{\text{OPP}}{\text{ADJ}} = \dfrac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$.

Whereas in the second quadrant, $\tan\left(\sec^{-1}(x)\right) = \frac{\text{OPP}}{\text{ADJ}} = \dfrac{\sqrt{x^2-1}}{-1} = -\sqrt{x^2-1}$.

Therefore our final answer is the piecewise function

$$\tan\left(\sec^{-1}(x)\right) = \begin{cases} \sqrt{x^2-1} & \text{if } x \text{ is positive} \\ -\sqrt{x^2-1} & \text{if } x \text{ is negative} \end{cases}$$

We will use this formula (as well as other simplifications in this section) later in the course. ✍

Exercises for Section 6.5

Simplify the given compositions.

1. $\tan\left(\sin^{-1}(x)\right) =$

2. $\tan\left(\cos^{-1}(x)\right) =$

3. $\tan\left(\tan^{-1}(x)\right) =$

4. $\sin\left(\cos^{-1}(x)\right) =$

5. $\sin\left(\sec^{-1}(x)\right) =$

6. $\sin\left(\sin^{-1}(x)\right) =$

7. $\cos\left(\sin^{-1}(x)\right) =$

8. $\cos\left(\tan^{-1}(x)\right) =$

9. $\cos\left(\sec^{-1}(x)\right) =$

10. $\sec\left(\sin^{-1}(x)\right) =$

11. $\sec\left(\cos^{-1}(x)\right) =$

12. $\sec\left(\tan^{-1}(x)\right) =$

6.6 Exercise Solutions for Chapter 6

Solutions for Section 6.1

1. $\sin^{-1}(-1) = -\frac{\pi}{2}$

3. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

5. $\sin^{-1}(1) = \frac{\pi}{2}$

7. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

Exercises for Section 6.3

1. $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

3. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

5. $\tan^{-1}(0) = 0$

7. $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{6}$

9. $\tan^{-1}(-1) = -\frac{\pi}{4}$

11. $\sec^{-1}(2) = \frac{\pi}{3}$

13. $\sec^{-1}\left(-\sqrt{2}\right) = -\frac{\pi}{4}$

15. $\sec^{-1}\left(\frac{2\sqrt{3}}{3}\right) = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

17. $\sin^{-1}\left(\sin(5\pi)\right) = \sin^{-1}(0) = 0$

19. $\tan^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right) = \tan^{-1}(1) = \frac{\pi}{4}$

21. $\sec^{-1}\left(\sec\left(\frac{5\pi}{4}\right)\right) = \sec^{-1}(\sqrt{2}) = \frac{\pi}{4}$

23. $\cos^{-1}\left(\cos(8\pi)\right) = \cos^{-1}(1) = 0$

25. Solve the equation $\tan^2(x) = \frac{1}{3}$.

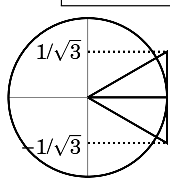
Taking square roots, this is

$$\tan(x) = \pm\frac{1}{\sqrt{3}}.$$

We can find all solutions that are in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ by taking $\tan^{-1}$ of this:

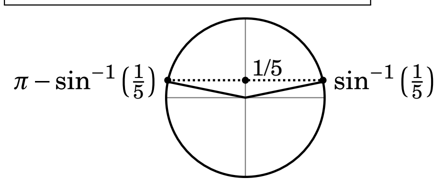
$$x = \tan^{-1}\left(\pm\frac{1}{\sqrt{3}}\right).$$

All solutions are: $\boxed{x = \tan^{-1}\left(\pm\frac{1}{\sqrt{3}}\right) + k\pi}$



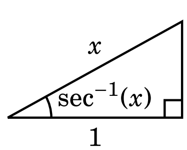
27. Find all solutions of $5x\sin(x) = x$ that are in the interval $[-\pi, \pi]$.

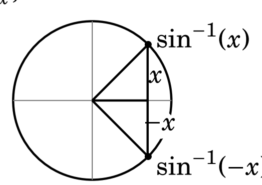
Clearly one solution is $x = 0$. Next suppose $x \neq 0$ and divide both sides of the equation by $5x$ to get $\sin(x) = \frac{1}{5}$, so $x = \sin^{-1}\left(\frac{1}{5}\right)$ is a solution in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the range of $\sin^{-1}$. There is another solution in $[-\pi, \pi]$, which is $x = \pi - \sin^{-1}\left(\frac{1}{5}\right)$. (See the diagram below.) Thus we have three solutions, $\boxed{0,\ \sin^{-1}\left(\tfrac{1}{5}\right) \text{ and } \pi - \sin^{-1}\left(\tfrac{1}{5}\right).}$



31. Explain why $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$.

Think of our standard picture for $\sec^{-1}(x)$, drawn below, left. From this triangle, notice that $\cos\left(\sec^{-1}(x)\right) = \frac{\text{ADH}}{\text{HYP}} = \frac{1}{x}$. In other words, $\sec^{-1}(x)$ is the angle $0 \le \theta \le \pi$ for which $\cos(\theta) = \frac{1}{x}$. Thus means $\sec^{-1}(x) = \theta = \cos^{-1}\left(\frac{1}{x}\right)$



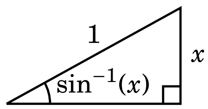


33. Explain why $\sin^{-1}(x) + \sin^{-1}(-x) = 0$ for all x in the domain of $\sin^{-1}$.

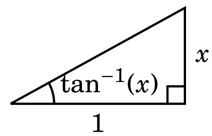
From the diagram for $\sin^{-1}(x)$ (above, right), note that $\sin^{-1}(-x) = -\sin^{-1}(x)$.

Exercises for Section 6.5.

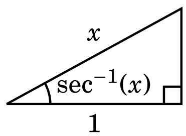
1. $\tan\left(\sin^{-1}(x)\right) = \frac{\text{OPP}}{\text{ADJ}} = \frac{x}{\sqrt{1-x^2}}$



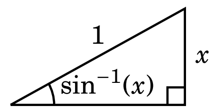
3. $\tan\left(\tan^{-1}(x)\right) = \frac{\text{OPP}}{\text{ADJ}} = x$



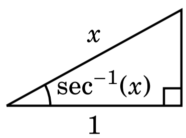
5. $\sin\left(\sec^{-1}(x)\right) = \frac{\text{OPP}}{\text{HYP}} = \frac{\sqrt{x^2-1}}{x}$



7. $\cos\left(\sin^{-1}(x)\right) = \frac{\text{ADJ}}{\text{HYP}} = \sqrt{1-x^2}$



9. $\cos\left(\sec^{-1}(x)\right) = \frac{\text{ADJ}}{\text{HYP}} = \frac{1}{x}$



11. $\sec\left(\cos^{-1}(x)\right) = \frac{\text{HYP}}{\text{ADJ}} = \frac{1}{x}$