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## Computing Limits Algebraically

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Chapter 7 introduced the limit as a mathematical method that can give slopes of tangent lines. Chapter 8 shifted the focus to the problem of evaluating limits in general, leaving the specific application to tangent slopes for later. We developed a set of *limit laws* that help us evaluate limits through algebraic operations. Let's check in briefly to see if the limit laws are getting us closer to our goal of finding tangent slopes.

In Chapter 7 we determined  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2$ , which gave the slope of the tangent to the graph of  $y = x^2 + 5$  at the point (1,6). If we were to blindly apply our new limit laws to this limit, we'd get

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{\lim_{x \rightarrow 1} (x^2 - 1)}{\lim_{x \rightarrow 1} (x - 1)} = \frac{1^2 - 1}{1 - 1} = \frac{0}{0}.$$

This makes no sense because  $\frac{0}{0}$  is undefined, and certainly not equal to 2, which we know is the correct value of the limit. The problem is that we applied Limit Law 5 incorrectly. It only works if the limit in the denominator is unequal to zero, because division by zero is impossible.

We can overcome this problem by factoring the numerator and canceling *before* applying the limit laws, as follows:

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) \frac{x - 1}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 1 + 1 = 2.$$

This situation is typical. The limits we will be most interested in—the ones that give slopes of tangent lines—tend to yield the form  $\frac{0}{0}$  initially. To solve them we need to *first* algebraically cancel the term that created the zero on the denominator, and *then* apply the limit laws. This chapter examines these kinds of limits.

The following examples illustrate various techniques and tricks for canceling a denominator that approaches zero.


**Example 9.1** Find  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 + x - 6}$ .

If we were to naively apply the limit laws we would get

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 + x - 6} = \frac{\lim_{x \rightarrow 2} (x^2 - 4)}{\lim_{x \rightarrow 2} (x^2 + x - 6)} = \frac{2^2 - 4}{2^2 + 2 - 6} = \frac{0}{0}.$$

But this makes no sense because  $\frac{0}{0}$  is not defined. In fact, Limit Law 5 does not apply because the limit on the denominator equals 0. As a second approach, we will try to factor and cancel before applying any limit laws.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 + x - 6} &= \lim_{x \rightarrow 2} \frac{(x+2)(x-2)}{(x+3)(x-2)} && \leftarrow \text{Factor top and bottom} \\ &= \lim_{x \rightarrow 2} \frac{x+2}{x+3} && \leftarrow \text{Cancel} \\ &= \frac{\lim_{x \rightarrow 2} (x+2)}{\lim_{x \rightarrow 2} (x+3)} && \leftarrow \text{Apply Limit Law 5} \\ &= \frac{2+2}{2+3} = \frac{4}{5} && \leftarrow \text{Answer!} \end{aligned}$$

Thus our final answer is  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 + x - 6} = \frac{4}{5}$ . 


The process used in this example is standard, and we summarize it as a general approach or “rule of thumb.”

**Rule of Thumb:** In working out a limit, try using the limit laws first. If applying them makes the denominator zero, try to algebraically cancel the part of the denominator that gives zero. Then proceed with the limit laws.

**Example 9.2** Find  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{x^4} + x}{x}$ .

If we attempted to solve this by first incorrectly applying Limit Law 5 from Chapter 8, the “answer” would be  $\frac{\sqrt[3]{0^4} + 0}{0} = \frac{0}{0}$ , which is not defined. The above Rule of Thumb suggests we should cancel the  $x$  in the denominator.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt[3]{x^4} + x}{x} &= \lim_{x \rightarrow 0} \frac{\sqrt[3]{x^3} \sqrt[3]{x} + x}{x} = \lim_{x \rightarrow 0} \frac{x \sqrt[3]{x} + x}{x} = \lim_{x \rightarrow 0} \frac{x(\sqrt[3]{x} + 1)}{x} \\ &= \lim_{x \rightarrow 0} (\sqrt[3]{x} + 1) = \sqrt[3]{0} + 1 = 1. \end{aligned}$$

Therefore the limit equals 1. 

In the previous two examples we simplified the limit by factoring and canceling. Doing this canceled the terms that were causing the limit to yield  $\frac{0}{0}$ , and put it into a form to which we could apply the limit laws.


This may not always be possible, at least not immediately. Next we will explore a very useful trick that can overcome this problem – a trick that can transform an expression into a factorable form. The idea is that we can always multiply a limit by 1 without changing its value. If  $g(x)$  is any algebraic expression, then we can multiply the limit by  $1 = \frac{g(x)}{g(x)}$ . If we are careful about selecting  $g(x)$ , this can simplify the limit dramatically. In the next example we multiply a limit by  $1 = \frac{x}{x}$ , and terms suddenly cancel.

**Example 9.3** Find  $\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{1 - x}$ .

In the limit  $x$  approaches 1, but simply plugging in  $x = 1$  gives  $\frac{\frac{1}{1} - 1}{1 - 1} = \frac{0}{0}$ . We would therefore hope to cancel the denominator,  $1 - x$ . But unfortunately we are blocked by the fact that we cannot factor out a  $1 - x$  from the top.

But notice that multiplying the numerator  $\frac{1}{x} - 1$  by  $x$  yields  $(\frac{1}{x} - 1)x = 1 - x$ , and this would cancel the denominator. But we are not allowed to randomly change the problem by inserting an  $x$  wherever we want it. However we *can* multiply by the limit by  $1 = \frac{x}{x}$  and use the top  $x$  to transform the numerator as we want it. We now carry out this plan.

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{1 - x} &= \lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{1 - x} \cdot \frac{x}{x} && \leftarrow \text{Multiply limit by } 1 = \frac{x}{x} \\ &= \lim_{x \rightarrow 1} \frac{(1 - x)}{(1 - x)x} && \leftarrow \text{Distribute } x \text{ on top} \\ &= \lim_{x \rightarrow 1} \frac{1}{x} && \leftarrow \text{Cancel the } (1 - x) \\ &= \frac{\lim_{x \rightarrow 1} 1}{\lim_{x \rightarrow 1} x} = \frac{1}{1} = 1 && \leftarrow \text{Apply limit laws} \end{aligned}$$

Thus the limit equals 1. Notice how after multiplying by  $\frac{x}{x}$ , the  $x$  on top did its job, and the  $x$  on the bottom just went along for the ride. 

Obviously, the success of using the trick of multiplying by  $1 = \frac{g(x)}{g(x)}$  relies completely on choosing the right  $g(x)$ . Fortunately these problems tend to fall into just two categories. One category involves eliminating a fraction, as we did above in changing  $\frac{1}{x} - 1$  to  $(\frac{1}{x} - 1)x = 1 - x$ . (After working a few exercises you will quickly come to recognize how and when to do this.) The other category eliminates radicals by *multiplying by the conjugate*.

Recall that the *difference of two squares formula* is  $(a - b)(a + b) = a^2 - b^2$ . The factors  $(a - b)$  and  $(a + b)$  are called *conjugates* of each other. That is,  $(a + b)$  is the conjugate of  $(a - b)$ , and  $(a - b)$  is the conjugate of  $(a + b)$ . The difference of two squares says that if we multiply  $a - b$  by its conjugate we get  $a^2 - b^2$ . This can be very useful when we wish to change an occurrence of  $a - b$  to  $a^2 - b^2$ . In the next problem we use it to change  $\sqrt{x} - 3$  to  $\sqrt{x^2} - 3^2 = x - 9$ . This is the first of many instances where the trick of “multiplying by the conjugate” helps us evaluate a limit.

**Example 9.4** Find  $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$ .

Our first impulse should be to try using the limit laws directly (as in the previous chapter), essentially plugging the value 9 that  $x$  approaches into the expression  $\frac{\sqrt{x} - 3}{x - 9}$ . But that results in the “answer”  $\frac{\sqrt{9} - 3}{9 - 9} = \frac{0}{0}$ , which makes no sense. We need to come up with another plan.

If only we could change the  $\sqrt{x} - 3$  to  $\sqrt{x^2} - 3^2 = x - 9$ , then it would cancel with the  $x - 9$  on the bottom, and the  $\frac{0}{0}$  problem would go away. We will do this by multiplying the limit by 1, in the form of the conjugate over itself.

$$\begin{aligned}
 \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} &= \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} && \leftarrow \text{Multiply by } 1 = \frac{\sqrt{x} + 3}{\sqrt{x} + 3}. \\
 &= \lim_{x \rightarrow 9} \frac{\sqrt{x}\sqrt{x} + \sqrt{x} \cdot 3 - 3\sqrt{x} - 3 \cdot 3}{(x - 9)(\sqrt{x} + 3)} && \leftarrow \text{FOIL top. Keep bottom} \\
 &= \lim_{x \rightarrow 9} \frac{\sqrt{x}\sqrt{x} - 3 \cdot 3}{(x - 9)(\sqrt{x} + 3)} && \leftarrow \text{factored because we hope} \\
 &= \lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} && \leftarrow \text{to cancel the } (x - 9). \\
 &= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} && \leftarrow \text{Simplify.} \\
 &= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} && \leftarrow \text{Simplify.} \\
 &= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} && \leftarrow \text{Cancel.}
 \end{aligned}$$

Now we have canceled the  $(x - 9)$  in the top with the  $(x - 9)$  in the bottom. These were what caused  $\frac{0}{0}$  when we first tried using the limit laws. Now that we have removed this obstacle we can finish using the limit laws.

$$\begin{aligned}
 &= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} \\
 &= \frac{1}{\sqrt{9} + 3} = \frac{1}{3 + 3} = \frac{1}{6} && \leftarrow \text{Apply limit laws.}
 \end{aligned}$$

We have now found our answer:  $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \frac{1}{6}$ .



Here is an alternate solution of Example 9.4. The denominator  $x - 9$  can be written as  $x - 9 = \sqrt{x^2} - 3^2$ , at least if  $x$  is positive (as it will be, if it's approaching 9). This factors as a difference of two squares, as  $x - 9 = \sqrt{x^2} - 3^2 = (\sqrt{x} + 3)(\sqrt{x} - 3)$ . Let's use this approach.

$$\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{\sqrt{x^2} - 3^2} = \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{(\sqrt{x} + 3)(\sqrt{x} - 3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}.$$

Please note that if  $x$  were approaching any number other than 9, then we wouldn't even have to cancel. In such a case the denominator would not approach zero, and we could use the limit laws directly. For example,

$$\lim_{x \rightarrow 4} \frac{\sqrt{x} - 3}{x - 9} = \frac{\lim_{x \rightarrow 4} (\sqrt{x} - 3)}{\lim_{x \rightarrow 4} (x - 9)} = \frac{\sqrt{4} - 3}{4 - 9} = \frac{1}{5}.$$

**Example 9.5** Find  $\lim_{x \rightarrow 7} \frac{\frac{1}{x} - \frac{1}{7}}{x - 7}$ .

We first try using the limit laws directly, plugging in 7. But that gives  $\frac{0}{0}$ , so we shift our strategy and try to cancel the  $x - 7$  in the denominator. Notice that the  $\frac{1}{x} - \frac{1}{7}$  on the top has a common denominator of  $7x$ . Multiplying the entire limit by  $1 = \frac{7x}{7x}$  should clear out the denominators. Let's try this.

$$\begin{aligned} \lim_{x \rightarrow 7} \frac{\frac{1}{x} - \frac{1}{7}}{x - 7} &= \lim_{x \rightarrow 7} \frac{\frac{1}{x} - \frac{1}{7}}{x - 7} \cdot \frac{7x}{7x} && \leftarrow \text{Multiply by } 1 = \frac{7x}{7x}. \\ &= \lim_{x \rightarrow 7} \frac{\frac{7x}{x} - \frac{7x}{7}}{(x - 7)7x} && \leftarrow \text{Distribute } 7x \text{ into top. Keep} \\ &= \lim_{x \rightarrow 7} \frac{7 - x}{(x - 7)7x} && \leftarrow \text{bottom factored because we} \\ &= \lim_{x \rightarrow 7} \frac{7 - x}{(x - 7)7x} && \leftarrow \text{hope to cancel the } (x - 7). \\ &= \lim_{x \rightarrow 7} \frac{-(x - 7)}{(x - 7)7x} && \leftarrow \text{Simplify (cancel top fractions).} \\ &= \lim_{x \rightarrow 7} \frac{-1}{7x} && \leftarrow \text{Factor } -1 \text{ out of top.} \\ &= \frac{-1}{7 \cdot 7} = -\frac{1}{49} && \leftarrow \text{Cancel the } (x - 7). \\ & && \leftarrow \text{Final answer!} \end{aligned}$$

We have now found our answer:  $\lim_{x \rightarrow 7} \frac{\frac{1}{x} - \frac{1}{7}}{x - 7} = -\frac{1}{49}$ .




So far our examples have used the variable  $x$ . Of course other variables can be used. The next two examples use the variable  $h$ . They are crafted to have the form of *derivatives*, an idea that will be introduced later.

**Example 9.6** Find  $\lim_{h \rightarrow 0} \frac{\frac{1}{(h-4)^2} - \frac{1}{16}}{h}$ .

If we were to plug in  $h = 0$ , we'd get  $\frac{0}{0}$ , so let's instead try to cancel the  $h$  in the denominator. The two fractions on top have a common denominator of  $16(h-4)^2$ , so we will try multiplying the entire limit by  $1 = \frac{16(h-4)^2}{16(h-4)^2}$  to clear out the denominators.

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{\frac{1}{(h-4)^2} - \frac{1}{16}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{(h-4)^2} - \frac{1}{16}}{h} \cdot \frac{16(h-4)^2}{16(h-4)^2} && \leftarrow \text{Multiply by } 1 = \frac{16(h-4)^2}{16(h-4)^2} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{16(h-4)^2}{(h-4)^2} - \frac{16(h-4)^2}{16}}{h \cdot 16(h-4)^2} && \leftarrow \text{Distribute on top} \\
 &= \lim_{h \rightarrow 0} \frac{16 - (h-4)^2}{h \cdot 16(h-4)^2} && \leftarrow \text{Cancel in top fractions} \\
 &= \lim_{h \rightarrow 0} \frac{16 - (h^2 - 8h + 16)}{h \cdot 16(h-4)^2} && \leftarrow \text{FOIL in top} \\
 &= \lim_{h \rightarrow 0} \frac{16 - h^2 + 8h - 16}{h \cdot 16(h-4)^2} && \leftarrow \text{Distribute negative} \\
 &= \lim_{h \rightarrow 0} \frac{-h^2 + 8h}{h \cdot 16(h-4)^2} && \leftarrow \text{Simplify top} \\
 &= \lim_{h \rightarrow 0} \frac{h(-h + 8)}{h \cdot 16(h-4)^2} && \leftarrow \text{Factor out } h \text{ on top} \\
 &= \lim_{h \rightarrow 0} \frac{(-h + 8)}{16(h-4)^2} && \leftarrow \text{Cancel the } h \\
 &= \frac{-0 + 8}{16(0-4)^2} && \leftarrow \text{Apply limit laws} \\
 &= \frac{1}{32} && \leftarrow \text{Final answer!}
 \end{aligned}$$


We have now found our answer:  $\lim_{h \rightarrow 0} \frac{\frac{1}{(h-4)^2} - \frac{1}{16}}{h} = \frac{1}{32}$ . 

This example showed every step. In practice you will likely skip and combine steps. But don't skip too many – it is there that mistakes creep in.

**Example 9.7** Find  $\lim_{h \rightarrow 0} \frac{\sqrt{5+h} - \sqrt{5}}{h}$ .

Substituting  $h = 0$ , would yield  $\frac{0}{0}$ , so we aim to cancel the  $h$  in the denominator. Taking our cue from Example 9.4 we will multiply by  $1 = \frac{\sqrt{5+h} + \sqrt{5}}{\sqrt{5+h} + \sqrt{5}}$ .

$$\begin{aligned}
 & \lim_{h \rightarrow 0} \frac{\sqrt{5+h} - \sqrt{5}}{h} \\
 = & \lim_{h \rightarrow 0} \frac{\sqrt{5+h} - \sqrt{5}}{h} \cdot \frac{\sqrt{5+h} + \sqrt{5}}{\sqrt{5+h} + \sqrt{5}} && \leftarrow \text{Multiply by } 1 = \frac{\sqrt{5+h} + \sqrt{5}}{\sqrt{5+h} + \sqrt{5}} \\
 = & \lim_{h \rightarrow 0} \frac{\sqrt{5+h}^2 + \sqrt{5+h}\sqrt{5} - \sqrt{5}\sqrt{5+h} - \sqrt{5}^2}{h(\sqrt{5+h} + \sqrt{5})} && \leftarrow \text{Distribute on top} \\
 = & \lim_{h \rightarrow 0} \frac{5+h-5}{h(\sqrt{5+h} + \sqrt{5})} && \leftarrow \text{Simplify} \\
 = & \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{5+h} + \sqrt{5})} && \leftarrow \text{Simplify} \\
 = & \lim_{h \rightarrow 0} \frac{1}{\sqrt{5+h} + \sqrt{5}} && \leftarrow \text{Cancel the } h \\
 = & \frac{1}{\sqrt{5+0} + \sqrt{5}} && \leftarrow \text{Apply limit laws} \\
 = & \frac{1}{2\sqrt{5}} && \leftarrow \text{Final answer!}
 \end{aligned}$$

We have now found our answer:  $\lim_{h \rightarrow 0} \frac{\sqrt{5+h} - \sqrt{5}}{h} = \frac{1}{2\sqrt{5}}$ . 

In summary, this chapter's main point is that many limits have form  $\lim_{x \rightarrow a} \frac{g(x)}{h(x)}$  where  $\frac{\lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} h(x)} = \frac{0}{0}$ , or the form  $\lim_{h \rightarrow 0} \frac{g(h)}{h}$  where  $\frac{\lim_{h \rightarrow 0} g(h)}{\lim_{h \rightarrow 0} h} = \frac{0}{0}$ .

To solve them we first cancel the term that makes the denominator approach 0. We have explored various techniques that accomplish this.

There are still some situations that our new techniques can't handle. For example,  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$  gives  $\frac{0}{0}$ , but there is no way to factor an  $x$  from the top to cancel the  $x$  on the bottom. We will learn how to handle this and other trigonometric limits in the next chapter. For now, it is vitally important that you get lots of practice working the exercises.

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**Exercises for Chapter 9**

1.  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$
2.  $\lim_{x \rightarrow 3} \frac{x^2 - 8x + 15}{x^2 - 2x - 3}$
3.  $\lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2}$
4.  $\lim_{x \rightarrow 1} \frac{\frac{1}{x} - 1}{x - 1}$
5.  $\lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h}$
6.  $\lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$
7.  $\lim_{x \rightarrow 3} \frac{\frac{1}{x^2} - \frac{1}{9}}{x - 3}$
8.  $\lim_{x \rightarrow -1} \frac{x^2 - 3x - 4}{x^2 + 5x + 4}$
9.  $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x^2 - x - 6}$
10.  $\lim_{x \rightarrow 2} \frac{x^2 - 7x + 10}{x^2 + x - 6}$
11.  $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$
12.  $\lim_{x \rightarrow -1} \frac{\frac{1}{x} + 1}{x + 1}$
13.  $\lim_{x \rightarrow 4} \frac{x - 4}{x^2 - 16}$
14.  $\lim_{x \rightarrow 3} \frac{x^2 + 7x - 30}{x - 3}$
15.  $\lim_{x \rightarrow 0^+} \frac{2x}{|x|}$
16.  $\lim_{x \rightarrow 3^+} \frac{3x - 9}{|x - 3|}$
17.  $\lim_{x \rightarrow 0^-} \frac{|x|}{4x} =$
18.  $\lim_{x \rightarrow 1^+} \frac{2x - 2}{|x - 1|}$
19.  $\lim_{x \rightarrow 1^-} \frac{2x - 2}{|x - 1|}$
20.  $\lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^2 + 4x - 5}$
21.  $\lim_{h \rightarrow 0} \frac{\sqrt{5+h} - \sqrt{5}}{h}$
22.  $\lim_{h \rightarrow 0} \frac{\sqrt{7-h} - \sqrt{7}}{h}$
23.  $\lim_{h \rightarrow 0} \frac{\sqrt{5-h} - \sqrt{5}}{h}$
24.  $\lim_{x \rightarrow 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3}$
25.  $\lim_{h \rightarrow 0} \frac{\frac{1}{6+h} - \frac{1}{6}}{h}$
26.  $\lim_{x \rightarrow 8} \frac{x - 2}{\sqrt{x}(\sqrt{x} - \sqrt{2})}$
27.  $\lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{x}(\sqrt{x} - \sqrt{2})}$
28.  $\lim_{x \rightarrow 1^+} \frac{\sqrt{2x}(x - 1)}{|x - 1|}$
29.  $\lim_{x \rightarrow 4^+} \frac{x^2 - 2x - 8}{|-x + 4|}$
30.  $\lim_{x \rightarrow 3^+} \frac{-x^2 - 2x + 15}{|-x + 3|}$
31.  $\lim_{x \rightarrow 3^-} \frac{-x^2 + x + 6}{|-x + 3|}$
32.  $\lim_{x \rightarrow 3^-} \frac{-x^2 + x + 6}{-x + 3}$



### Solutions for Chapter 9

1.  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4$
3.  $\lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{\sqrt{x^2 + 12} - 4}{x - 2} \cdot \frac{\sqrt{x^2 + 12} + 4}{\sqrt{x^2 + 12} + 4} = \lim_{x \rightarrow 2} \frac{x^2 + 12 - 16}{(x - 2)(\sqrt{x^2 + 12} + 4)}$   
 $= \lim_{x \rightarrow 2} \frac{x^2 - 4}{(x - 2)(\sqrt{x^2 + 12} + 4)} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{(x - 2)(\sqrt{x^2 + 12} + 4)}$   
 $= \lim_{x \rightarrow 2} \frac{x + 2}{\sqrt{x^2 + 12} + 4} = \frac{2 + 2}{\sqrt{2^2 + 12} + 4} = \frac{1}{2}$
5.  $\lim_{h \rightarrow 0} \frac{\sqrt{4 + h} - 2}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{4 + h} - 2}{h} \cdot \frac{\sqrt{4 + h} + 2}{\sqrt{4 + h} + 2} = \lim_{h \rightarrow 0} \frac{(4 + h) - 4}{h(\sqrt{4 + h} + 2)}$   
 $= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{4 + h} + 2)} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{4 + h} + 2} = \frac{1}{\sqrt{4 + 0} + 2} = \frac{1}{4}$
7.  $\lim_{x \rightarrow 3} \frac{\frac{1}{x^2} - \frac{1}{9}}{x - 3} = \lim_{x \rightarrow 3} \frac{\frac{1}{x^2} - \frac{1}{9}}{x - 3} \cdot \frac{9x^2}{9x^2} = \lim_{x \rightarrow 3} \frac{9 - x^2}{(x - 3)9x^2} = \lim_{x \rightarrow 3} \frac{(3 - x)(3 + x)}{(x - 3)9x^2}$   
 $= \lim_{x \rightarrow 3} \frac{-(3 + x)}{9x^2} = \frac{-(3 + 3)}{9 \cdot 3^2} = \frac{-6}{81} = -\frac{2}{27}$
9.  $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x^2 - x - 6} = \lim_{x \rightarrow -2} \frac{(x - 2)(x + 2)}{(x - 3)(x + 2)} = \lim_{x \rightarrow -2} \frac{x - 2}{x - 3} = \frac{-2 - 2}{-2 - 3} = \frac{4}{5}$
11.  $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \lim_{x \rightarrow 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \lim_{x \rightarrow 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{6}$
13.  $\lim_{x \rightarrow 4} \frac{x - 4}{x^2 - 16} = \lim_{x \rightarrow 4} \frac{x - 4}{(x - 4)(x + 4)} = \lim_{x \rightarrow 4} \frac{1}{x + 4} = \frac{1}{4 + 4} = \frac{1}{8}$
15.  $\lim_{x \rightarrow 0^+} \frac{2x}{|x|} = \lim_{x \rightarrow 0^+} \frac{2x}{x} = 2$  ( $|x| = x$  when  $x$  is positive)
17.  $\lim_{x \rightarrow 0^-} \frac{|x|}{4x} = \lim_{x \rightarrow 0^-} \frac{-x}{4x} = -\frac{1}{4}$  ( $|x| = -x$  when  $x$  is negative)
19.  $\lim_{x \rightarrow 1^-} \frac{2x - 2}{|x - 1|} = \lim_{x \rightarrow 1^-} \frac{2(x - 1)}{-(x - 1)} = -2$  ( $|x - 1| = -(x - 1)$  when  $x - 1$  is negative)
21.  $\lim_{h \rightarrow 0} \frac{\sqrt{5 + h} - \sqrt{5}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{5 + h} - \sqrt{5}}{h} \cdot \frac{\sqrt{5 + h} + \sqrt{5}}{\sqrt{5 + h} + \sqrt{5}} = \lim_{h \rightarrow 0} \frac{(5 + h) - 5}{h(\sqrt{5 + h} + \sqrt{5})}$   
 $= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{5 + h} + \sqrt{5})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{5 + h} + \sqrt{5}} = \frac{1}{\sqrt{5 + 0} + \sqrt{5}} = \frac{1}{2\sqrt{5}}$

$$\begin{aligned}
 23. \quad \lim_{h \rightarrow 0} \frac{\sqrt{5-h} - \sqrt{5}}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{5-h} - \sqrt{5}}{h} \cdot \frac{\sqrt{5-h} + \sqrt{5}}{\sqrt{5-h} + \sqrt{5}} = \lim_{h \rightarrow 0} \frac{(5-h) - 5}{h(\sqrt{5-h} + \sqrt{5})} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{h(\sqrt{5-h} + \sqrt{5})} = \lim_{h \rightarrow 0} \frac{-1}{\sqrt{5-h} + \sqrt{5}} = \frac{-1}{\sqrt{5+0} + \sqrt{5}} = -\frac{1}{2\sqrt{5}}
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \lim_{h \rightarrow 0} \frac{\frac{1}{6+h} - \frac{1}{6}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{6+h} - \frac{1}{6}}{h} \cdot \frac{6(6+h)}{6(6+h)} = \lim_{h \rightarrow 0} \frac{6 - (6+h)}{h6(6+h)} = \lim_{h \rightarrow 0} \frac{-h}{h6(6+h)} \\
 &= \lim_{h \rightarrow 0} \frac{-1}{6(6+h)} = \frac{-1}{6(6+0)} = -\frac{1}{36}
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x}(\sqrt{x}-\sqrt{2})} &= \lim_{x \rightarrow 2} \frac{\sqrt{x^2} - \sqrt{2^2}}{\sqrt{x}(\sqrt{x}-\sqrt{2})} = \lim_{x \rightarrow 2} \frac{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}{\sqrt{x}(\sqrt{x}-\sqrt{2})} \\
 &= \lim_{x \rightarrow 2} \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x}} = \frac{\sqrt{2} + \sqrt{2}}{\sqrt{2}} = 2
 \end{aligned}$$

$$29. \quad \lim_{x \rightarrow 4^+} \frac{x^2 - 2x - 8}{|-x + 4|} = \lim_{x \rightarrow 4^+} \frac{(x-4)(x+2)}{-(-x+4)} = \lim_{x \rightarrow 4^+} (x+2) = 6$$

$$31. \quad \lim_{x \rightarrow 3^-} \frac{-x^2 + x + 6}{|-x + 3|} = \lim_{x \rightarrow 3^-} \frac{(-x+3)(x+2)}{-x+3} = \lim_{x \rightarrow 3^-} (x+2) = 5$$