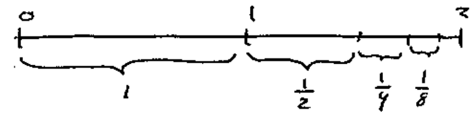
Chapter 10 **Sequences and Infinite Series**

In this chapter we will examine infinite sums.

Example: $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2$



Such considerations lead to important mathematical facts.

Examples:
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

$$\cos x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$\frac{e^x}{x^3} = \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{2x} + \frac{1}{3!} + \frac{x}{4!} + \frac{x^2}{5!} + \cdots$$

There are two basic ideas we are going to examine in Ch 10

A) Infinite Series $a_0 + a_1 + a_2 + a_3 + \cdots$ (∞ sum of #'s)

B) Infinite Sequences $a_0 \quad a_1 \quad a_2 \quad a_3 \quad \ldots$ (∞ list of #'s)

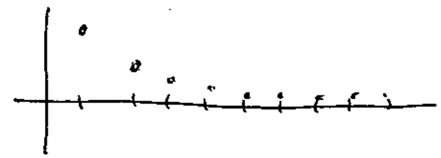
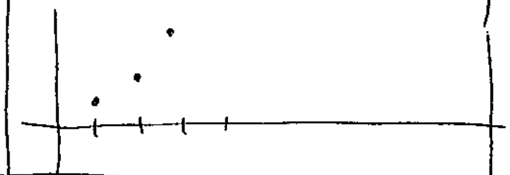
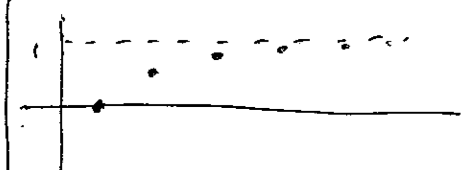
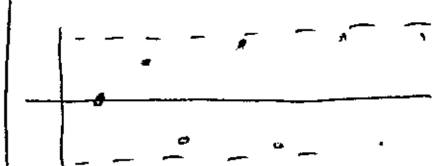
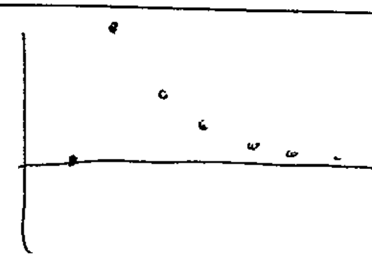
Infinite sequences are more basic and That's where we start

Section 10.2 Infinite sequences

Definition

An infinite sequence is an infinite list $a_0 \; a_1 \; a_2 \ldots$ of numbers in a specified order

Examples

Sequence	n^{th} term	Brace Notation	Graph	Limit
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \cdots$	$f(n) = \frac{1}{n}$	$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$		$\lim_{n\to\infty} \frac{1}{n} = 0$ sequence converges to 0
$1 \ 4 \ 9 \ 16 \ 25 \ldots$	$f(n) = n^2$	$\left\{n^2\right\}_{n=1}^{\infty}$		$\lim_{n\to\infty} n^2 = \infty$ sequence diverges to ∞
$0 \ \frac{1}{2} \ \frac{2}{3} \ \frac{3}{4} \ \frac{4}{5} \ldots$	$f(n) = \frac{n-1}{n}$	$\left\{\frac{n-1}{n}\right\}_{n=1}^{\infty}$		$\lim_{n\to\infty} \frac{n-1}{n} = 1$ sequence converges to
$-0 \ +\frac{1}{2} \ -\frac{2}{3} \ +\frac{3}{4} \ -\frac{4}{5} \ldots$	$f(n) = (-1)^n \frac{n-1}{n}$	$\left\{(-1)^n \frac{n-1}{n}\right\}_{n=1}^{\infty}$		$\lim_{n\to\infty} (-1)^n \frac{n-1}{n}$ DNE sequence diverges
$\frac{\ln 1}{1} \ \frac{\ln 2}{2} \ \frac{\ln 3}{3} \ldots$		$\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$		$\lim_{n\to\infty} \frac{\ln(n)}{n} =$ $\lim_{n\to\infty} \frac{1/n}{1} = 0$ sequence converges to

Note: Many of our familiar limit rules don't apply directly here because n takes on only integer values. Also terms such as $(-1)^{n+1}$ never occured previously because $(-1)^x$ is not continuous or even well defined for all x. We need to be careful about exactly what the limit means

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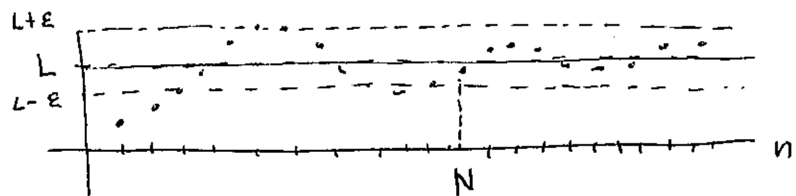
Definition A sequence $\{a_n\}$ converges to L, written $\lim_{n\to\infty} a_n = L$, if given any $\varepsilon > 0$ there is an $N > 0$ such that $L - \varepsilon < a_n < L + \varepsilon$ for all $n \geq N$.



If for some $\varepsilon > 0$, no such N exists, the sequence diverges.

Using this definition, you can prove numerous results, which we will simply accept as fact.

Theorem Suppose $\{a_n\} = a_1\ a_2\ a_3 \cdots$ $\{b_n\} = b_1\ b_2\ b_3 \cdots$ $\{c_n\} = c, c, c, \cdots$ are sequences. From these we get other sequences

$\{a_n + b_n\} = a_1+b_1\ \ a_2+b_2\ \ a_3+b_3 \cdots$ $\{a_n b_n\} = a_1 b_1\ \ a_2 b_2\ \ a_3 b_3 \cdots$

$\{\frac{a_n}{b_n}\} = \frac{a_1}{b_1}\ \ \frac{a_2}{b_2}\ \ \frac{a_3}{b_3} \cdots$ etc.

If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

(a) $\lim_{n\to\infty} c_n = c$

(b) $\lim_{n\to\infty} \{a_n \pm b_n\} = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

(c) $\lim_{n\to\infty} \{a_n b_n\} = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$

(d) $\lim_{n\to\infty} \{\frac{a_n}{b_n}\} = \lim_{n\to\infty} a_n / \lim_{n\to\infty} b_n$

(e) $\lim_{n\to\infty} \{c a_n\} = c \lim_{n\to\infty} \{a_n\}$

Theorem If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

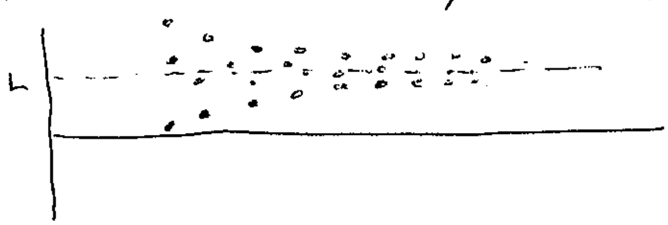
Ex $\left\{ \frac{\sqrt{n}+1}{n+1} \right\}_{n=1}^{\infty}$ $\quad \lim\limits_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = \lim\limits_{n\to\infty} \frac{\frac{\sqrt{n}}{n}+\frac{1}{n}}{1+\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{\frac{1}{\sqrt{n}}+\frac{1}{n}}{1+\frac{1}{n}}$

$$= \frac{\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n}}+\frac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)} = \frac{0+0}{1+0} = \boxed{0}$$

Ex $\left\{ (-1)^n \frac{\sqrt{n}+1}{n+1} \right\}_{n=1}^{\infty}$ $\quad$ **Note** $\lim\limits_{n\to\infty} \left| (-1)^n \frac{\sqrt{n}+1}{n+1} \right| =$

$\lim\limits_{n\to\infty} \frac{\sqrt{n}+1}{n+1} = 0$, so $\boxed{\lim\limits_{n\to\infty} (-1)^n \frac{\sqrt{n}+1}{n+1} = 0}$

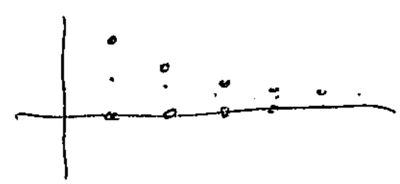
Squeezing Theorem Suppose $\{a_n\}$ $\{b_n\}$ $\{c_n\}$ are sequences for which $a_n \le b_n \le c_n$ for sufficiently large n. If $\{a_n\}$ and $\{b_n\}$ have limit L, then $\{b_n\}$ has limit L also



Application $\left\{ \frac{2^n}{n!} \right\}_{n=1}^{\infty}$ What is limit?

$$0 \le \frac{2^n}{n!} = \frac{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot \;\cdots\; 2\cdot 2\cdot 2}{n(n-1)(n-2)(n-3)(n-4)\cdots 3\cdot 2\cdot 1} \le \frac{2(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 2}{n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1} = \frac{4}{n}$$

$\{0\}$ $\quad \left\{\frac{2^n}{n!}\right\}$ $\quad \left\{\frac{4}{n}\right\}$ $\qquad 0 \le \frac{2^n}{n!} \le \frac{4}{n}$



Then $\lim\limits_{n\to\infty} \frac{2^n}{n!} = 0$

Sequences defined recursively $\quad$ **Ex** $\sqrt{6}$ $\quad \sqrt{6+\sqrt{6}}$ $\quad \sqrt{6+\sqrt{6+\sqrt{6}}}$ $\quad \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}}$ $\ldots$

$a_n = \sqrt{6+a_{n-1}}$ $\qquad$ What is the limit?

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \sqrt{6+a_{n-1}}$

$L = \sqrt{6+L}$

$L^2 = 6+L$

$L^2 - L - 6 = 0$

$(L-3)(L+2) = 0$

L=3