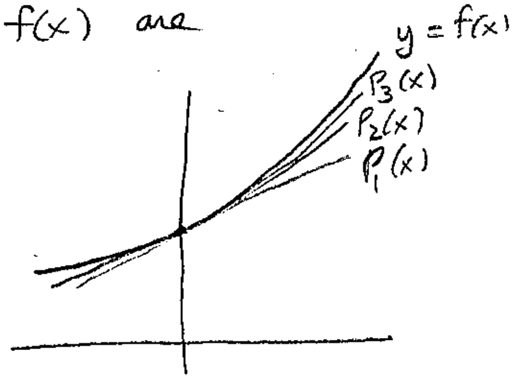
§11.1 Continued

Recall: Given a function $f(x)$ its Maclaurin series is

$$p(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

The Maclaurin polynomials for $f(x)$ are



$$P_1(x) = f(0) + f'(0)x$$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\vdots$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Fact $\quad P_n^{(k)}(0) = f^{(k)}(0) \quad$ for $\quad 0 \le k \le n$

Example Maclaurin series for $f(x) = e^x$ is

$$p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

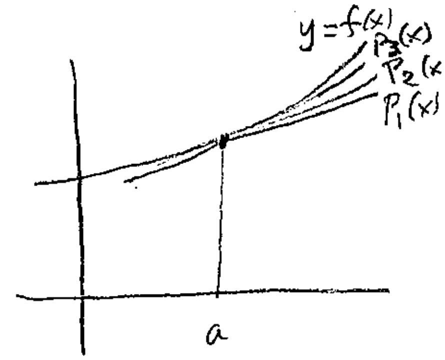
(see previous lecture)

The Maclaurin series and polynomials are a special case of a more general construction.

Definition Given a function $f(x)$, and a number a, the Taylor series for $f(x)$ centered at a is

$$p(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f'(a)}{2!}(x-a)^2 + \cdots$$

The Taylor polynomials for $f(x)$ are



$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$\vdots$$

$$\vdots$$

Fact $p_n^{(k)}(a) = f^{(k)}(0)$ for $0 \le k \le n$

$\boxed{\text{Note: Maclaurin series / polynomials for } f(x) \text{ with } a=0 \text{ are just its Taylor series / polynomials}}$

Example Find Taylor series for $f(x) = \ln(x)$ centered at $a=1$.

$$\left.\begin{aligned} f(x) &= \ln(x) \\ f'(x) &= \frac{1}{x} \\ f''(x) &= \frac{-1}{x^2} \\ f'''(x) &= \frac{2}{x^3} \\ f^{(4)}(x) &= \frac{-3\cdot 2}{x^4} \\ f^{(5)}(x) &= \frac{4\cdot 3\cdot 2}{x^5} \\ &\vdots \end{aligned}\right\} \quad \begin{aligned} f^{(k)}(x) &= \frac{(-1)^{k-1}(k-1)!}{x^k} \\ f^{(k)}(1) &= (-1)^{k-1}(k-1)! \end{aligned}$$

Taylor series

$$p(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!}(x-1)^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k(k-1)!}(x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$$

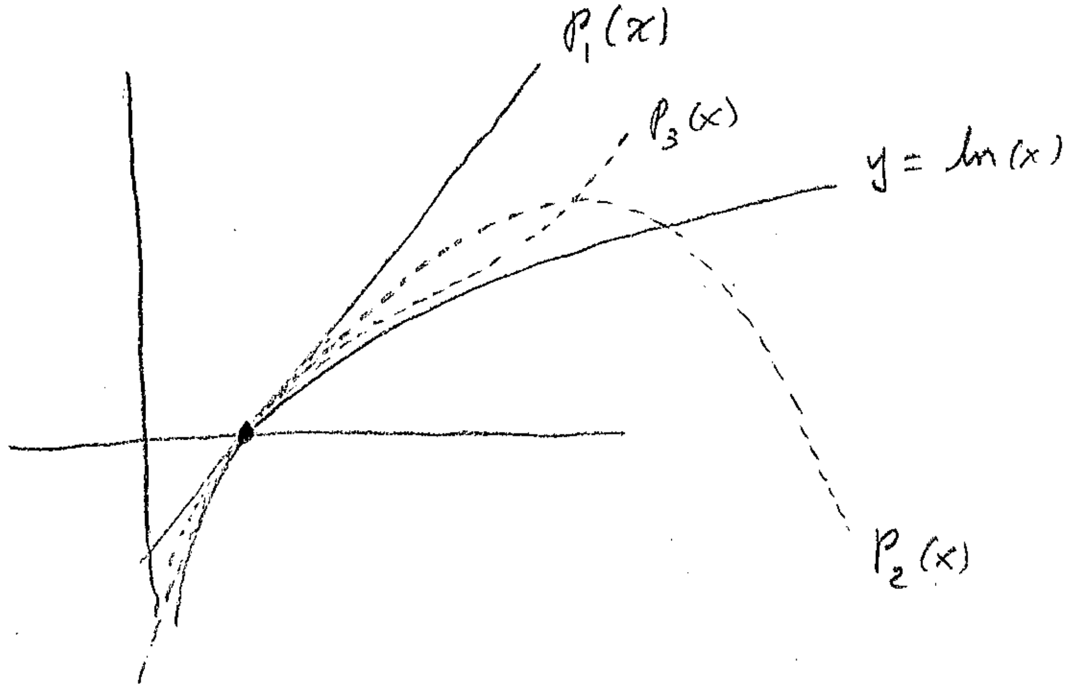
$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$

Taylor Polynomials centered at $a=1$ for $\ln(x)$

$$P_1(x) = x-1$$

$$P_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

$$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

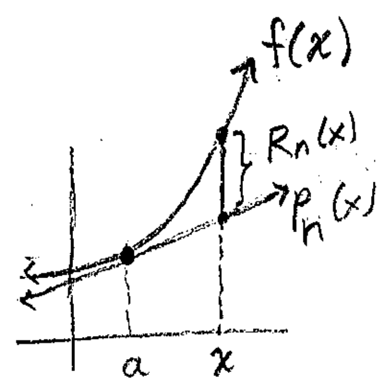


The larger n, the better $P_n(x)$ approximates the graph of $\ln(x)$.

<u>Question</u> In general how good is the approximation? To begin to answer this question, we make a definition.

<u>Definition</u>

If $P_n(x)$ is a Taylor polynomial for $f(x)$, the <u>remainder</u> is $R_n(x) = f(x) - P_n(x)$.



Theorem 11.1 Taylors Remainder Theorem

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x,

Theorem 11.2

If for some n, $|f^{(n+1)}(c)| \le M$ for all $a \le c \le x$,

then $\left|R_n(x)\right| \le M \frac{|x-a|^{n+1}}{(n+1)!}$

Example Let $f(x) = \sin(x)$

$f'(x) = \cos(x)$ $\quad f'(0) = 1$

$f''(x) = -\sin(x)$ $\quad f''(0) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(0) = -1$

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(0) = 0$

$$P(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}}_{P_9(x)} - \cdots$$

Note $|f^{10}(c)| \le 1$

$$\therefore |R_n(x)| \le \left|\frac{1 \cdot x^{10}}{10!}\right| = \frac{x^{10}}{3628800}$$

For reasonably small values of x, (say $-1 \le x \le 1$)

$R_n(x)$ is very small!