§11.3 Taylor Series

Recall: Given a function $f(x)$,

- Taylor Series for $f(x)$ centered at a: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

- Maclaurin Series for $f(x)$: $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$

Maclaurin series is just the Taylor series centered at $a=0$

Taylors Remainder Theorem

$$f(x) = \underbrace{\frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f^{(1)}(a)}{1!}(x-1)^1 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)}$$

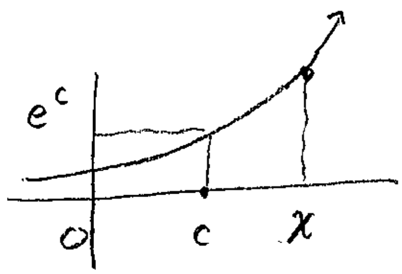
where c is a number between a and x. The number c may depend on a, x and n.

Consequence (pretty obvious): Given an x, the Taylor series for $f(x)$ converges to $f(x)$ provided that

$$\lim_{n\to\infty} R_n(x) = 0.$$

<u>Example</u> $f(x) = e^x$

$$e^x = \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}}_{P_n(x)} + \underbrace{\frac{e^c}{(n+1)!}x^{n+1}}_{R_n(x)}$$



$\Leftarrow$ <u>Note</u>: $e^c \leq e^x$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{e^c}{(n+1)!}x^{n+1} \leq \lim_{n\to\infty} \frac{e^x x^{n+1}}{(n+1)!}$$

$$e^x \lim_{n\to\infty} \frac{x \cdot x \cdot x \cdot x \cdots x}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (n+1)}$$

$$e^x \lim_{n\to\infty} \underbrace{\frac{x\, x\, x \quad x}{1\, 2\, 3 \quad k}}_{< x^k} \underbrace{\frac{x}{k+1}\frac{x}{k+2}\frac{x}{k+3}\cdots\frac{x}{n}}_{<1} \cdot \frac{x}{n+1} = 0$$

$k \leq x \leq k+1$

approaches 0

Therefore $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$ with convergence on $(-\infty, \infty)$.

Summary of some Taylor Series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{on } (-\infty, \infty)$$

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{on } (-\infty, \infty)$$

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{on } (-\infty, \infty)$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \quad \text{on } (-1, 1)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \text{on } (-1, 1)$$

$$\int \frac{1}{1+x}\,dx = \ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \quad \text{on } (-1, 1)$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} \cdots \quad \text{on } (-1, 1)$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad \text{on } (-1, 1)$$

Ex $\quad x^5 e^x = x^5\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = x^5 + x^6 + \frac{x^7}{2!} + \frac{x^8}{3!} + \cdots$

Ex $\quad e^{x^3} = 1 + x^3 + \frac{(x^3)^2}{2!} + \frac{(x^3)^3}{3!} + \cdots = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \frac{x^{12}}{4!} + \cdots$

Ex $\quad e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots = 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \cdots$

Binomial Series

Consider writing a Maclaurin series for $f(x)=\sqrt{1+x}=(1+x)^{\frac{1}{2}}$
More generally, consider doing this for $f(x)=(1+x)^m$
This leads to what is called the binomial series. .

Let's start with a concrete example: $f(x)=(1+x)^3$

$$\begin{aligned}
f^{(0)}(x) &= (1+x)^3 & f^{(0)}(0) &= 1 \\
f^{(1)}(x) &= 3(1+x)^2 & f^{(1)}(0) &= 3 \\
f^{(2)}(x) &= 6(1+x) & f^{(2)}(0) &= 6 \\
f^{(3)}(x) &= 6 & f^{(3)}(0) &= 6 \\
f^{(4)}(x) &= 0 & f^{(4)}(0) &= 0 \\
f^{(5)}(x) &= 0 & f^{(5)}(0) &= 0 \\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{aligned}
(1+x)^3 &= f^{(0)}(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\
&= 1 + 3x + \frac{6}{2}x^2 + \frac{6}{6}x^3 + 0x^4 + 0x^5 + \cdots \\
&= 1 + 3x + 3x^2 + x^3
\end{aligned}$$

Ex Find the Maclaurin series for $f(x)=(1+x)^p$

$$\begin{aligned}
f^{(0)}(x) &= (1+x)^p & f^{(0)}(0) &= 1 \\
f^{(1)}(x) &= p(1+x)^{p-1} & f^{(1)}(0) &= p \\
f^{(2)}(x) &= p(p-1)(1+x)^{p-2} & f^{(2)}(0) &= p(p-1) \\
f^{(3)}(x) &= p(p-1)(p-2)(1+x)^{p-3} & f^{(3)}(0) &= p(p-1)(p-2) \\
&\vdots & &\vdots \\
f^{(k)}(x) &= p(p-1)(p-2)(p-3)\cdots(p-k+1)(1+x)^{p-k} & f^{(k)}(0) &= p(p-1)(p-2)\cdots(p-k+1)
\end{aligned}$$

$$(x+1)^p = \frac{f(0)x^0}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \cdots$$

$$(x+1)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \frac{p(p-1)(p-2)(p-3)}{4!}x^4 + \cdots$$

$$(x+1)^p = 1 + \sum_{k=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k$$

Ex $\sqrt{x+1} =$

$$(x-1)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$

Notation $\dfrac{p(p-1)(p-2)\cdots(p-k+1)}{k!} = \dbinom{p}{k}$

Binomial Theorem

$$(1+x)^p = 1 + \sum_{k=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-k+1)}{k!} x^k$$

$$= 1 + \sum_{k=1}^{\infty} \binom{p}{k} x^k$$

and this converges for $|x| < 1$