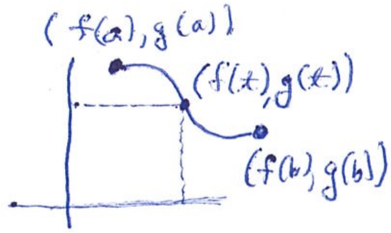
§12.3 Calculus in Polar Coordinates

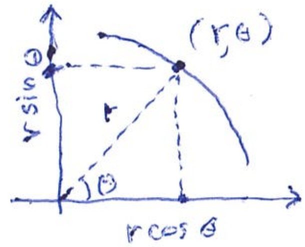


To begin, recall the following facts about a parametric curve $\left.\begin{array}{l} x = f(t) \\ y = g(t) \end{array}\right\} \; a \le t \le b.$

① Slope at $(f(t), g(t))$ is $m = \dfrac{g'(t)}{f'(t)} = \dfrac{dy/dt}{dx/dt}$.

② Length is $\displaystyle\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Next note that any polar curve $r = f(\theta)$ is easily converted to a parametric form, with parameter θ.

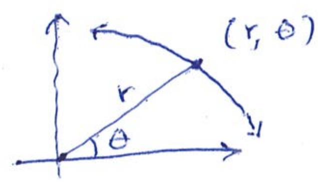


$$\boxed{\begin{array}{l}\text{Polar} \\ r = f(\theta)\end{array}} \longrightarrow \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \longrightarrow \boxed{\begin{array}{l}\text{Parametric} \\ \begin{cases} x = f(\theta)\cos(\theta) \\ y = f(\theta)\sin(\theta) \end{cases}\end{array}}$$

Putting all this together leads to two quick polar properties.

The slope of $r = f(\theta)$ at (r, θ) is

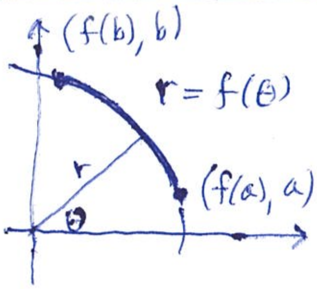
$$m = \frac{dy/dt}{dx/dt} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos\theta}{f'(\theta)\cos(\theta) - f(\theta)\sin\theta}$$



Length of $r = f(\theta)$, $a \le \theta \le b$ is $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

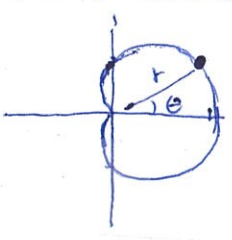
$$= \int_a^b \sqrt{(f'(t)\cos\theta - f(t)\sin\theta)^2 + (f'(t)\sin\theta + f(t)\cos\theta)^2}\, d\theta$$

$$= \cdots = \boxed{\int_a^b \sqrt{(f'(t))^2 + (f(t))^2}\, dt}$$



Example

$r = 1 + \cos\theta$



$$m = \frac{-\sin\theta\sin\theta + (1+\cos\theta)\cos\theta}{-\sin\theta\cos\theta - (1+\cos\theta)\sin\theta} = \frac{\cos^2\theta - \sin^2\theta + \cos\theta}{-\sin\theta - 2\sin\theta\cos\theta}$$

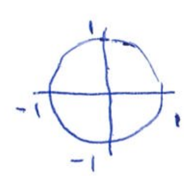
(Note e.g. $m = 0$ for $\theta = \frac{\pi}{3}$ which matches the graph)

Ex Circumference of circle $r = 1$.

$$L = \int_0^{2\pi} \sqrt{(f'(t))^2 + (f(t))^2}\, dt$$

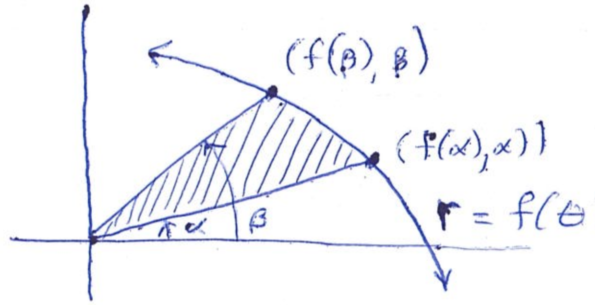
$$= \int_0^{2\pi} \sqrt{0 + 1}\, dt$$

$$= \int_0^{2\pi} dt = [t]_0^{2\pi} = \boxed{2\pi \text{ units}}$$



AREA

Problem: Find the area:

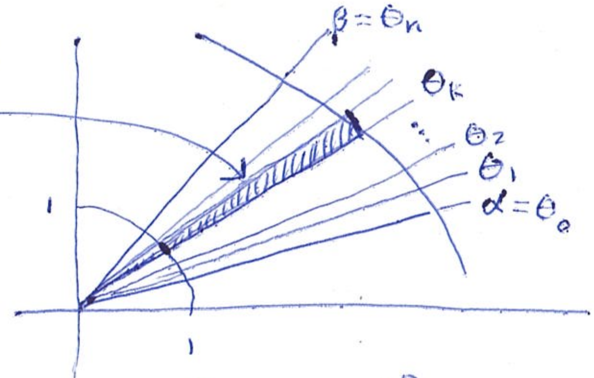


Solution $\Delta\theta = \dfrac{\beta - \alpha}{n}$

$\theta_0 = \alpha, \quad \theta_K = \alpha + K\Delta\theta, \quad \dots \quad \theta_n = \beta$

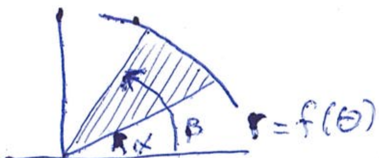
Area of triangle # K is $\frac{1}{2} bh = \frac{1}{2} f(\theta_K)\Delta\theta\, f(\theta_K)$

$= \frac{1}{2}\left(f(\theta_K)\right)^2 \Delta\theta$

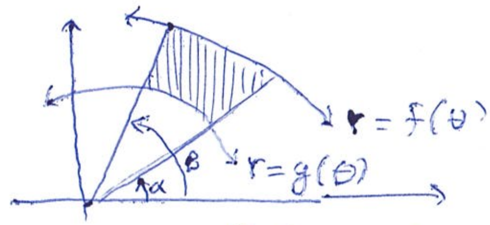


$$A = \lim_{n\to\infty} \sum_{K=1}^{n} \frac{1}{2}\left(f(\theta_K)\right)^2 \Delta\theta = \frac{1}{2}\int_\alpha^\beta \left(f(\theta)\right)^2 d\theta.$$

Formulas

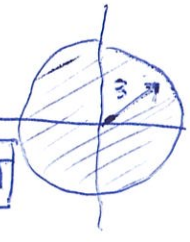


$$A = \frac{1}{2}\int_\alpha^\beta f(\theta)^2\, d\theta$$



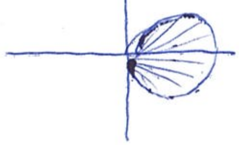
$$A = \frac{1}{2}\int_\alpha^\beta \left(f(\theta)^2 - g(\theta)^2\right) d\theta$$

Example Find the area of the circle $r = 3$



Ans $A = \frac{1}{2}\int_0^{2\pi} 3^2\, d\theta = \frac{9}{2}\int_0^{2\pi} d\theta = \frac{9}{2}\left[\theta\right]_0^{2\pi} = \boxed{9\pi}$

Example Find area inside $r = \sqrt{\cos\theta}$ between $\alpha = -\frac{\pi}{2}$ and $\beta = \frac{\pi}{2}$

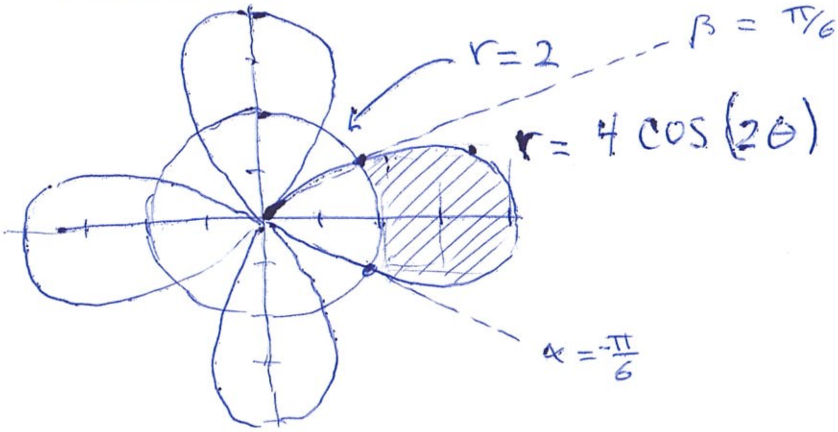


$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \sqrt{\cos\theta}^{\,2}\, d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \frac{1}{2}\Big[-\sin\theta\Big]_{-\pi/2}^{\pi/2} = \frac{1}{2}\left(-\sin\left(-\frac{\pi}{2}\right) - \left(-\sin\left(\frac{\pi}{2}\right)\right)\right)$$

$= \frac{1}{2}(1+1) = 1$ sq unit.

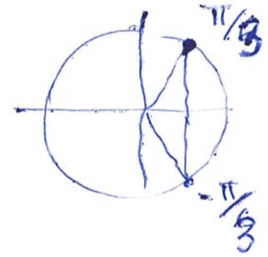
Example Find the area:



First we need to find the intersection points.

They happen where $2 = 4\cos(2\theta)$

$$\frac{1}{2} = \cos(2\theta)$$



$$2\theta = \pm\frac{\pi}{3} + 2\pi k$$

$$\theta = \pm\frac{\pi}{6} + \pi k \quad \text{when } k = 0, \pm 1, \pm 2, \cdots$$

For our portion of the graph (shaded) we get $\alpha = -\frac{\pi}{6}$, $\beta = \frac{\pi}{6}$

$$A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(4\cos(2\theta)\right)^2 - 2^2 \, d\theta = 1/2 \int_{-\pi/6}^{\pi/6} 16\cos^2(2\theta) - 4 \, d\theta$$

$u = 2\theta$
$du = 2d\theta$

$$= 1/2 \int_{-\pi/6}^{\pi/6} 2\left(4\cos^2(2\theta) - 1\right) 2 \, d\theta = \int_{-\pi/3}^{\pi/3} \left(4\cos^2(u) - 1\right) du$$

$$= \Big[4\left(\frac{u}{2} + \frac{1}{2}\cos u \sin u\right) - u\Big]_{-\pi/3}^{\pi/3}$$

$$= \Big[u + 2\cos u \sin u\Big]_{-\pi/3}^{\pi/3} = \left(\left(\frac{\pi}{3} + 2\cos\frac{\pi}{3}\sin\frac{\pi}{3}\right) - \left(\frac{-\pi}{3} + 2\cos\frac{-\pi}{3}\sin\frac{-\pi}{3}\right)\right)$$

$$= \left(\frac{\pi}{3} + 2\,\frac{1}{2}\,\frac{\sqrt{3}}{2} + \frac{\pi}{3} - 2\,\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)\right) = \boxed{\frac{2\pi}{3} + \sqrt{3} \text{ sq units}}$$