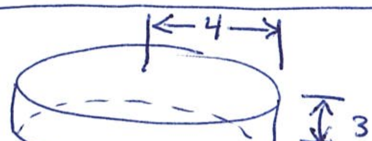
Section 6.3 Volumes by Slicing

Today we're going to look at volumes of solids. Some volumes are easy to compute, so that's where we're going to start.

It's easy to find the volume of a right cylinder

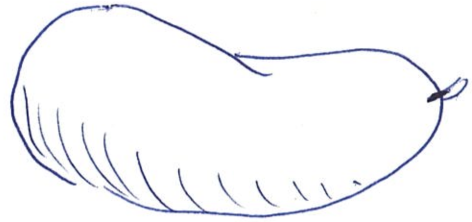
Volume = (area of base)(height) = Ah.

Ex

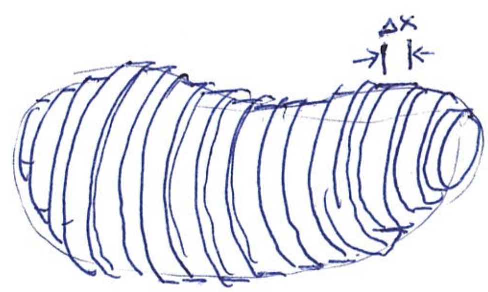


$$V = (\pi \cdot 4^2)\, 3 = \boxed{84\pi \text{ cubic units}}$$

Now lets think about how we could extend this idea to find volumes of more complicated shapes. Here right cylinders will play a role analogous to the rectangles in area problems.

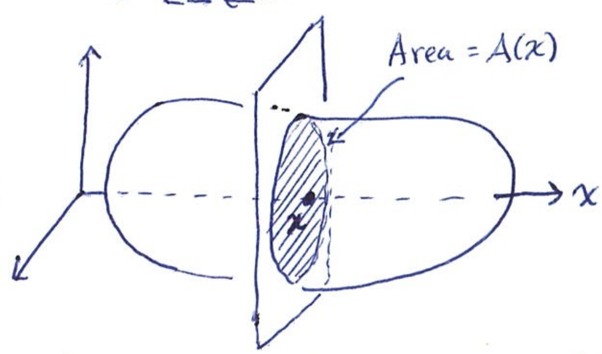


The basic idea is to approximate the shape with thin right cylinders, add up their volumes



To do this we need to find the area of the base of each cylinder

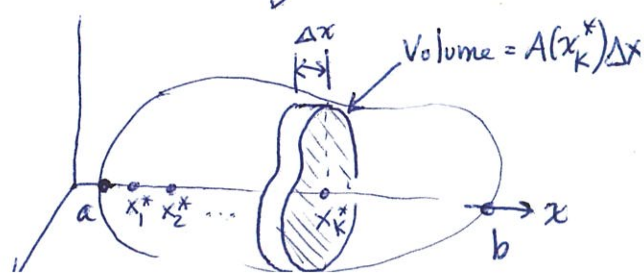
Let $A(x)$ = area of cross-section at x



$$V \approx \sum_{k=1}^{n} A(x_k^*)\,\Delta x$$

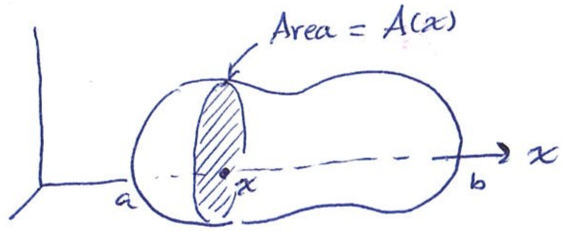
$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k^*)\,\Delta x$$

$$= \int_a^b A(x)\,dx$$

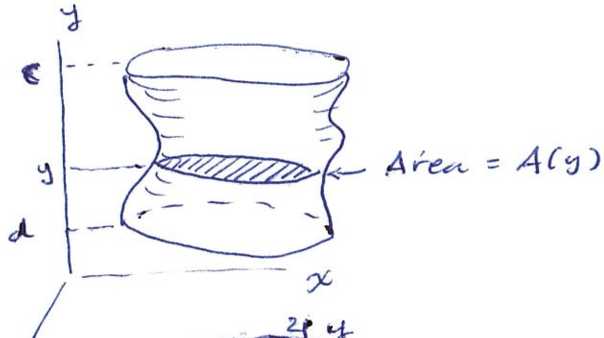


Conclusion

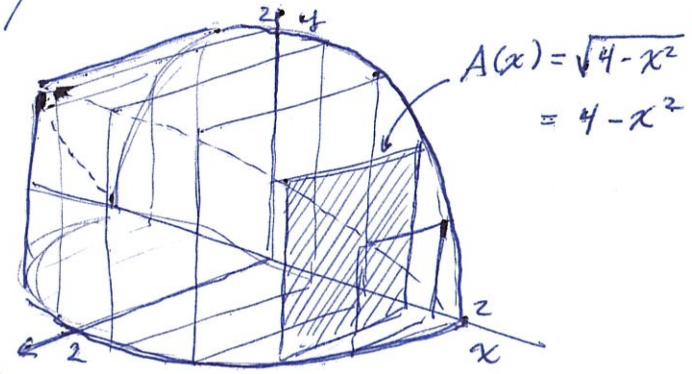
If a solid has cross-sectional area $A(x)$ at x, then the volume of the solid contained between a and b is $\int_a^b A(x)\,dx$.



Likewise you could have a solid bounded by y-values of c and d. If the cross-sectional area at y is $A(y)$, the volume of the solid is $\int_c^d A(y)\,dy$.



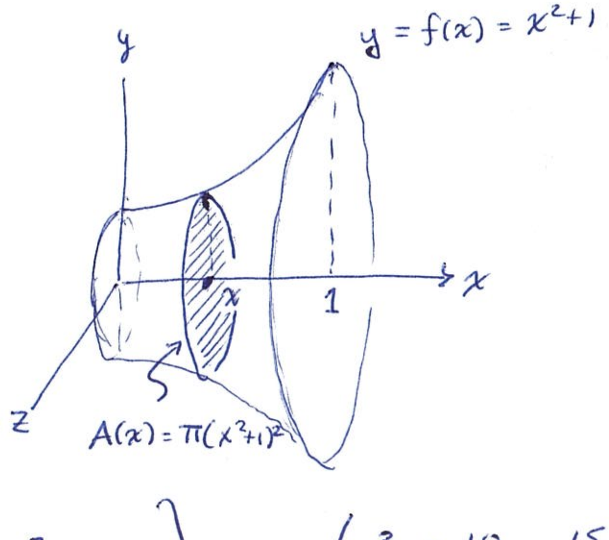
Example Find the volume of this shape. Cross-sections perpindicular to the x-axis are squares.



$$V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} (4-x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(4\cdot 2 - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$= \frac{32}{3}$ cubic units.

Example: $V = \int_0^1 A(x)\,dx$



$$= \int_0^1 \pi (x^2+1)^2\,dx$$

$$= \pi \int_0^1 (x^4 + 2x^2 + 1)\,dx$$

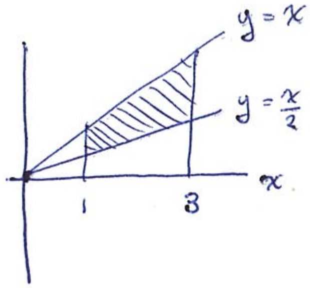
$$= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^1 = \pi\left(\frac{1}{5} + \frac{2}{3} + 1\right) = \pi\left(\frac{3}{15} + \frac{10}{15} + \frac{15}{15}\right)$$

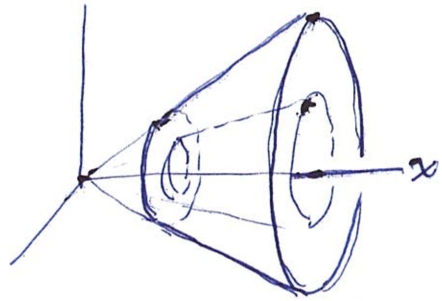
$= \frac{28\pi}{15}$ cubic units ≈ 5.86 cubic units.

Volume by "washers"

Ex

Rotate this around the x-axis. What's the volume?

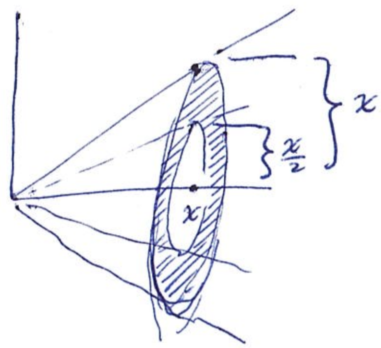




Look at cross-sectional area:

Area of big circle πx^2

Area of small circle $\pi\left(\frac{x}{2}\right)^2$



Area of ring: $\pi x^2 - \pi\left(\frac{x}{2}\right)^2$

$= \pi x^2 - \pi \frac{x^2}{4} = \frac{3\pi}{4} x^2$

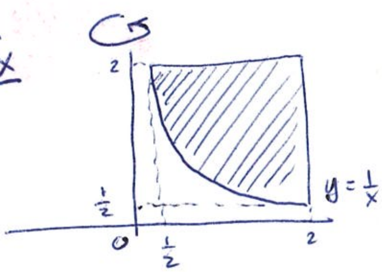
The ring is sometimes called a "washer."

$\underline{\text{Volume}}$: $V = \int_1^3 A(x)\,dx = \int_1^3 \frac{3\pi}{4} x^2\,dx$

$= \frac{3\pi}{4}\int_1^3 x^2\,dx = \frac{3\pi}{4}\left[\frac{x^3}{3}\right]_1^3 = \frac{3\pi}{4}\left[\frac{3^3}{3} - \frac{1^3}{3}\right]$

$= \frac{3\pi}{4}\left[9 - \frac{1}{3}\right] = \frac{3\pi}{4}\,\frac{26}{3} = \boxed{\frac{13}{2}\pi \text{ cubic units}}$

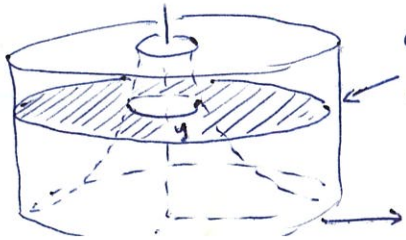
Ex



$V = \int_{1/2}^2 A(y)\,dy = \int_{1/2}^2 \left(4\pi - \frac{\pi}{y^2}\right) dy = \pi \int_{1/2}^2 \left(4 - \frac{1}{y^2}\right) dy$

$= \pi\left[4y + \frac{1}{y}\right]_{1/2}^2 = \pi\left[\left(4\cdot 2 + \frac{1}{2}\right) - \left(4\cdot\frac{1}{2} + \frac{1}{1/2}\right)\right] = \pi\left[8 + \frac{1}{2} - 2 - 2\right]$

$= \pi\left[4 + \frac{1}{2}\right] = \boxed{\frac{9\pi}{2} \text{ cubic units}}$



Cross-section at y:

$A(y) = \pi 2^2 - \pi\left(\frac{1}{y}\right)^2$

$= 4\pi - \frac{\pi}{y^2}$