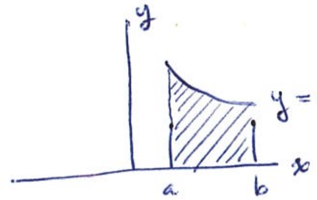
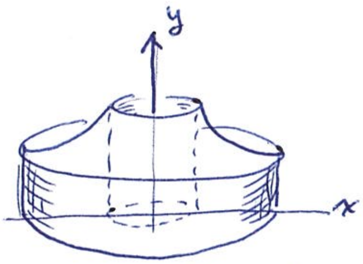
Section 6.4 Volumes by Cylindrical Shells

Here's a method that sometimes works when cross-sectional area is problematic.

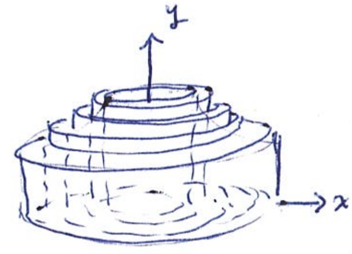
Basic Idea:



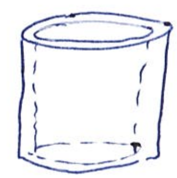
Rotate around y axis



What is the volume?



Approximate with "shells".

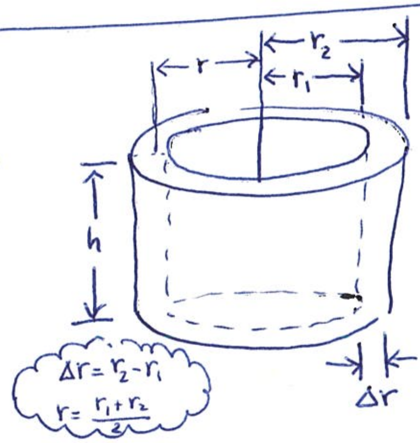


$$V \approx \sum_{k=1}^{n} \begin{pmatrix}\text{Volume of} \\ \text{shell \# } k\end{pmatrix}$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} \begin{pmatrix}\text{Volume of} \\ \text{shell \# } k\end{pmatrix} = \int_{?}^{?} ?\, dx$$

In order to carry out this program, we need to find a formula for the volume of a shell.

$$V = 2\pi r h \Delta r.$$



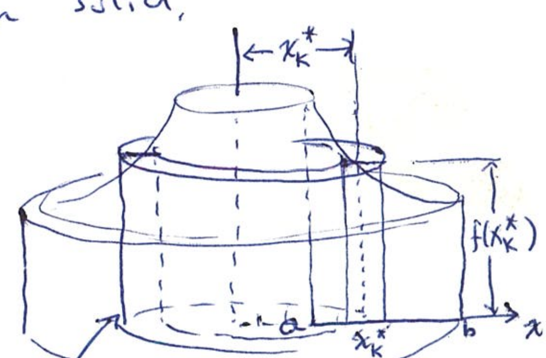
$$\begin{aligned} V &= \begin{pmatrix}\text{area of} \\ \text{base}\end{pmatrix} h \\ &= (\pi r_2^2 - \pi r_1^2)h \\ &= \pi(r_2^2 - r_1^2)h \\ &= \pi(r_2 + r_1)(r_2 - r_1)h \\ &= 2\pi \frac{r_2 + r_1}{2}(r_2 - r_1)h \\ &= 2\pi r \Delta r h \end{aligned}$$

Now we can work out the volume of our solid.

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*) \Delta x$$

$$= \int_a^b 2\pi x f(x)\, dx$$



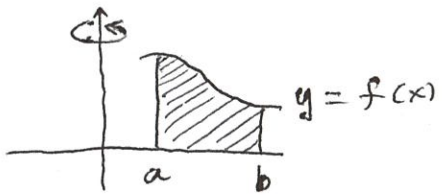
k^{th} shell

$r = x_k^*$ $h = f(x_k^*)$ $\Delta r = \Delta x$

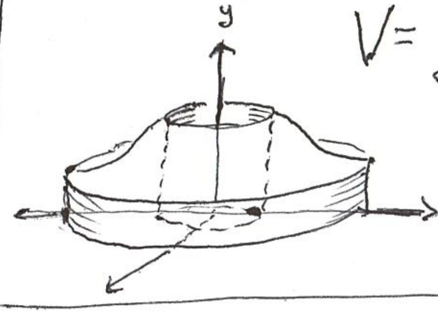
$V = 2\pi x_k^* f(x_k^*) \Delta x$

Conclusion:

Rotate this around y-axis

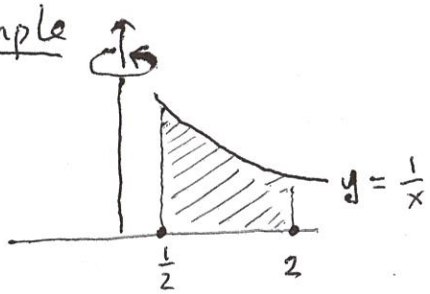


$\Longrightarrow$

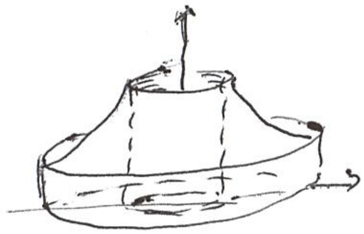


$$V = \int 2\pi x\, f(x)\, dx$$

Example



$\Rightarrow$



$$V = \int_{1/2}^{2} 2\pi x \frac{1}{x}\, dx$$

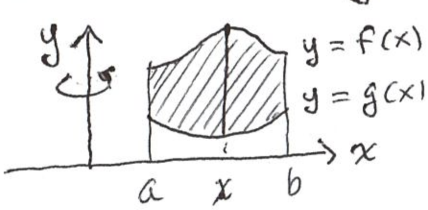
$$= \int_{1/2}^{2} 2\pi\, dx$$

$$= \Big[2\pi x\Big]_{1/2}^{2} = \boxed{3\pi \text{ cubic units}}$$

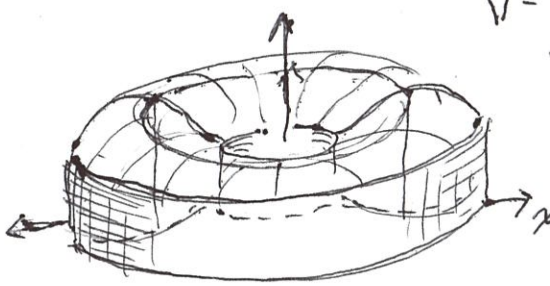
In general: $V = \int 2\pi \underset{\substack{\uparrow \\ \text{shell radius}}}{x}\; \underbrace{f(x)}_{\substack{\uparrow \\ \text{shell height}}}\, dx$

Conclusion

Rotate this around y-axis

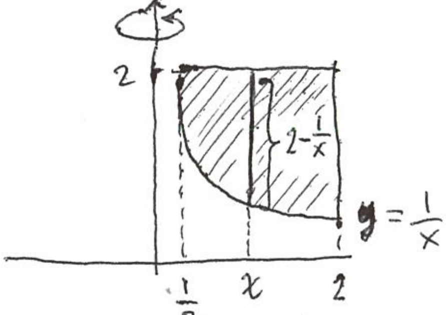


$\Rightarrow$

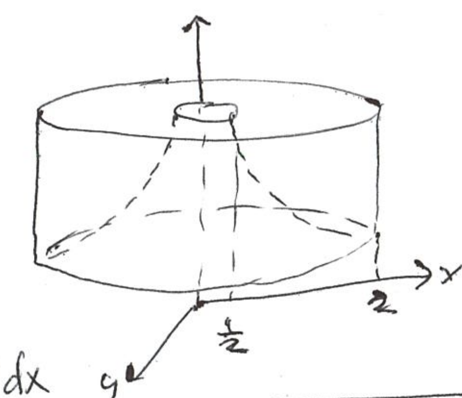


$$V = \int_a^b 2\pi x \big(f(x) - g(x)\big)\, dx$$

Ex



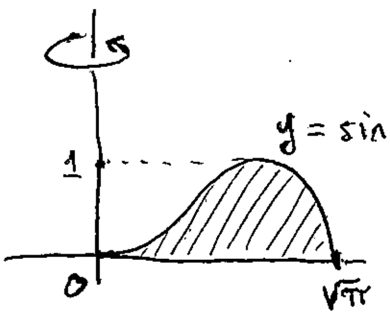
$\Longrightarrow$

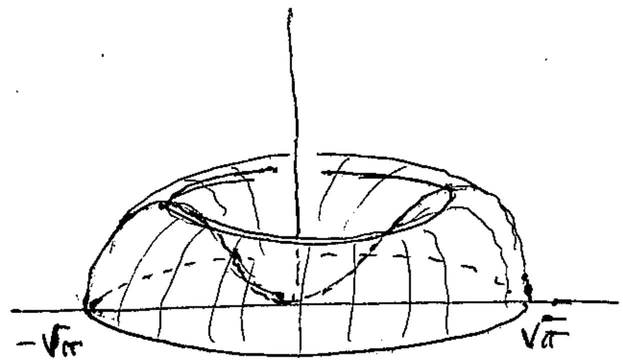


$$V = \int_{1/2}^{2} 2\pi x\left(2 - \frac{1}{x}\right) dx = 2\pi \int_{1/2}^{2} (2x - 1)\, dx$$

$$= 2\pi \Big[x^2 - x\Big]_{1/2}^{2} = \cdots = \boxed{\frac{9\pi}{2} \text{ cubic units}}$$

Ex





$$V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\,dx = \pi \int_0^{\sqrt{\pi}} \sin(x^2)\, 2x\, dx$$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$$= \pi \int_{0^2}^{\sqrt{\pi}^2} \sin(u)\,du$$

$$= \pi \Big[-\cos(u) \Big]_0^{\pi}$$

$$= \pi \Big(-\cos(\pi) - (-\cos(0)) \Big)$$

$$= \pi \big(-(-1) + 1 \big)$$

$$= \boxed{2\pi \text{ cubic units}}$$