Chapter 8 Integration Techniques

Section 8.1 Basic Approaches

Our goal in Chapter 8 is to learn more integration formulas and techniques.
Recall our list of basic integration formulas, given below.
Our first task is to expand it to include integration formulas for tan, cot, sec and csc.
Then we will look at some additional examples. (The formula for $\int \ln|u|\,dx$ will come in Section 8.2.)

Integration Formulas

$\int c\,dx = cx + C$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + C$
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$	$\int \csc^2(ax)\,dx = -\frac{1}{a}\cot(ax) + C$
$\int \frac{1}{x}\,dx = \ln|x| + C$	$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + C$
$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$	$\int \csc(ax)\cot(ax)\,dx = -\frac{1}{a}\csc(ax) + C$
$\int b^x\,dx = \frac{1}{\ln(b)}b^x + C$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C \quad (a>0)$
$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$	$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$	$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C \quad (a>0)$
$\int \tan(ax)\,dx = \frac{1}{a}\ln|\sec(ax)| + C$	
$\int \cot(ax)\,dx = \frac{1}{a}\ln|\sin(ax)| + C$	derived on next page
$\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(ax) + \tan(ax)| + C$	
$\int \csc(ax)\,dx = -\frac{1}{a}\ln|\csc(ax) + \cot(ax)| + C$	
$\int \ln|x|\,dx =$	

Substitution Rule

If $u = g(x)$, then $\int f(g(x))\,g'(x)\,dx = \int f(u)\,dx$.

If $u = g(x)$, then $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,dx$.

Let's get right to work

Ex $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{1}{\cos(x)} \sin(x)\,dx$

$u = \cos(x)$
$\frac{du}{dx} = -\sin(x)$
$-du = \sin(x)dx$

$= \int \frac{-1}{u}\,du = -\ln|u| + C = \ln\left|\frac{1}{u}\right| + C$

$= \ln\left|\frac{1}{\cos(x)}\right| + C = \boxed{\ln|\sec(x)| + C}$

New formula: $\boxed{\begin{array}{l} \int \tan(x)\,dx = \ln|\sec(x)| + C \\ \int \tan(ax)\,dx = \frac{1}{a}\ln|\sec(ax)| + C \end{array}}$

Ex $\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx = \int \frac{1}{\sin(x)} \cos(x)\,dx$

$u = \sin(x)$
$du = \cos(x)dx$

$= \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\sin(x)| + C}$

New formula: $\boxed{\begin{array}{l} \int \cot(x)\,dx = \ln|\sin(x)| + C \\ \int \cot(ax)\,dx = \frac{1}{a}\ln|\sin(ax)| + C \end{array}}$

Ex $\int \sec(x)\,dx = \int \sec(x) \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\,dx$

$= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\,dx = \int \frac{1}{\sec(x) + \tan(x)}(\sec(x)\tan(x) + \sec^2(x))$

$= \int \frac{1}{\sec(x) + \tan(x)}(\sec(x)\tan(x) + \sec^2(x))\,dx = \int \frac{1}{u}\,du$

$= \ln|u| + C = \boxed{\ln|\sec(x) + \tan(x)| + C}$

New Formula: $\boxed{\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(x) + \tan(x)| + C}$

Now try this: $\int \csc(x)\,dx = \cdots = -\ln|\csc(x)+\cot(x)| + C$

Formula: $\boxed{\int \csc(ax)\,dx = \cdots = -\frac{1}{a}\ln|\csc(ax)+\cot(ax)| + C}$

Examples

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

Ex $\int 3x\tan(x^2)\,dx = 3\int \tan(x^2)\,x\,dx$

$= 3\int \tan(u)\,\frac{1}{2}\,du = \frac{3}{2}\int \tan(u)\,du$

$= \frac{3}{2}\ln|\sec(u)| + C = \boxed{\frac{3}{2}\ln|\sec(x^2)| + C}$

$$\begin{array}{r} x+1 \\ x-1\,\overline{)\,x^2+0x+2} \\ \underline{x^2-x} \\ x+2 \\ \underline{x-1} \\ 3 \end{array}$$

Ex $\int_2^4 \frac{x^2+2}{x-1}\,dx = \int_2^4 x+1+\frac{3}{x-1}\,dx$

$= \left[\frac{x^2}{2}+x+3\ln|x-1|\right]_2^4$

$= \left(\frac{4^2}{2}+4+3\ln|4-1|\right) - \left(\frac{2^2}{2}+2+3\ln|2-1|\right)$

$= 8+4+3\ln|3| - 2 - 2 - 3\ln|1| = \boxed{8+3\ln|3|}$

$u = \sqrt{2}x$

$\frac{du}{dx} = \sqrt{2}$

$dx = \frac{1}{\sqrt{2}}\,du$

Ex $\int \frac{5}{3+2x^2}\,dx = 5\int \frac{dx}{\sqrt{3}^2+(\sqrt{2}x)^2}$

$= 5\int \frac{1}{\sqrt{3}^2+u^2}\,\frac{1}{\sqrt{2}}\,du$

$= \frac{5}{\sqrt{2}}\int \frac{1}{\sqrt{3}^2+u^2}\,du = \frac{5}{\sqrt{2}}\,\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$

$= \boxed{\frac{5}{\sqrt{6}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{3}}\right) + C}$

Algebra Review Completing the square.

Ex x^2+6x ⟵ Goal: make this a perfect square

$$= x^2+6x+\left(\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2$$

$$= x^2+6x+9-9$$

$$= (x+3)(x+3)-9$$

$$= (x+3)^2-9$$

Rule x^2+bx (when coefficient of x is 1)

$$= x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$$

$$= \left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$$

Rule ax^2+bx (when coefficient of x is a)

$$= ax^2+bx+\frac{b^2}{4a}-\frac{b^2}{4a}$$

$$= \left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)-\frac{b^2}{4a}$$

$$= \left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2-\frac{b^2}{4a}$$

Ex $$\int\frac{1}{x^2+5x+7}\,dx = \int\frac{1}{7-\frac{25}{4}+\left(x^2+5x+\frac{25}{4}\right)}\,dx$$

$$= \int\frac{1}{\frac{3}{4}+\left(x+\frac{5}{2}\right)^2}\,dx$$

$u = x+5/2$

$\frac{du}{dx} = 1$

$du = dx$

$$= \int\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2+u^2}\,du$$

$$= \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right)+C = \boxed{\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+5}{\sqrt{3}}\right)+C}$$