Section 8.2 Integration By Parts

Let's face it. We still are unable to compute most antiderivatives. We balk at something as innocent as $\int \ln(x)\, x^2\, dx$. Now we come to a technique that allows us to compute this and many others. It's called <u>integration by parts</u>. It comes from the product rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

$$\int \left(f(x)g'(x) + g(x)f'(x)\right)dx = f(x)g(x)$$

$$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

This expression is useful when one of the integrals is easier to evaluate than the other. Consider:

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)f'(x)\,dx$$

$$\int \ln x \;\; x^2\,dx = \ln(x)\,\frac{x^3}{3} - \int \frac{x^3}{3}\,\frac{1}{x}\,dx = \boxed{\frac{\ln(x)\,x^3}{3} - \frac{x^3}{9} + C}$$

It's customary to use the following notation:

$u = f(x)$ $\quad v = g(x)$

$du = f'(x)dx$ $\quad dv = g'(x)dx$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$\int u\,dv = u\,v - \int v\,du$$

Integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

Here's how we usually work these problems.

$$\int \underbrace{\ln(x)}_{u}\, \underbrace{x^2 dx}_{dv} = \ln(x)\frac{x^3}{3} - \int \frac{x^3}{3}\frac{1}{x}\,dx = \frac{\ln(x)x^3}{3} - \int \frac{x^2}{3}\,dx$$

$$= \boxed{\frac{\ln(x)x^3}{3} - \frac{x^3}{9} + C}$$

$u = \ln x \quad dv = x^2 dx$

$du = \frac{1}{x}dx \quad v = \frac{x^3}{3}$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = \boxed{x\sin x + \cos x + C}$$

$u = x \quad dv = \cos x\,dx$

$du = dx \quad v = \sin x$

Notice that success revolves around making the right choice of u.
Sometimes you will make the wrong choice..

$$\int x\cos x\,dx = \cos x\frac{x^2}{2} - \underbrace{\int \frac{x^2}{2}(-\sin x)\,dx}_{\text{more complicated.}}$$

$u = \cos x \quad dv = x\,dx$

$du = -\sin x\,dx \quad v = \frac{x^2}{2}dx$

When this happens (and it will) start over again. After awhile you learn to look a few steps ahead, and you will make the right choices, most of the time.

$$\int xe^{2x}dx = x\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}dx = \boxed{\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C}$$

$u = x \quad dv = e^{2x}dx$

$du = dx \quad \frac{1}{2}e^{2x} = v$

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}(x) - \frac{1}{2}\int (1-x^2)^{\frac{1}{2}}\,2x\,dx$$

$$= \boxed{x\sin^{-1}(x) + \sqrt{1-x^2} + C}$$

$u = \sin^{-1}(x) \quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx \quad v = x$

$$\int \ln(x)\,dx = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - \int dx = \boxed{x\ln x - x + C}$$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad V = x$

New Formula! $\boxed{\int \ln|x|\,dx = x\ln|x| - x + C}$

Sometimes you'll have to use integration by parts twice.

$$\int x^2 e^{3x}\,dx = x^2 \tfrac{1}{3} e^{3x} - \int \tfrac{1}{3} e^{3x}\, 2x\, dx$$

$u = x^2 \qquad dV = e^{3x}\,dx$

$du = 2x\,dx \qquad V = \tfrac{1}{3}e^{3x}$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int x e^{3x}\,dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left(\frac{x e^{3x}}{3} - \int \tfrac{1}{3} e^{3x}\,dx\right)$$

$u = x \qquad dv = e^{3x}\,dx$

$du = dx \qquad V = \tfrac{1}{3}e^{3x}$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left(\frac{x e^{3x}}{3} - \frac{1}{9} e^{3x}\right)$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2x e^{3x}}{9} + \frac{2}{27} e^{3x} + C$$

Some problems for you:

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$$\int x^3 \cos(x^2)\,dx = uv - \int v\,du = \frac{x^2 \sin(x^2)}{2} - \int \tfrac{1}{2}\sin(x^2)\, 2x\,dx$$

$u = x^2 \qquad dv = \cos(x^2)\, x\,dx$

$du = 2x\,dx \qquad v = \tfrac{1}{2}\int \cos(x^2)\, 2x\,dx$

$\qquad\qquad = \tfrac{1}{2}\sin(x^2)$

$$= \frac{x^2 \sin(x^2)}{2} + \frac{1}{2}\cos(x^2) + C$$