§ 8.6 Integration Strategies

Be ready to think on your feet and decide which integration techniques may work for any given problem. Sometimes several different techniques will work equally well. Sometimes one you think will work will not work. Just keep trying. This section is really just integration practice, using the methods discussed in § 8.1 – § 8.5. We'll just do a series of examples.

Ex $\displaystyle\int \frac{1-x}{1+x}\,dx = \int \frac{1-(u-1)}{u}\,du = \int \frac{2-u}{u}\,du = \int \frac{2}{u} - \frac{u}{u}\,du$

$u = 1+x$
$du = dx$
$x = u-1$

$\displaystyle = \int \left(\frac{2}{u} - 1\right) du = 2\ln|u| - u + C$

$= 2\ln|1+x| - (1+x) + C = \boxed{2\ln|1+x| - x + C}$

Ex $\displaystyle\int \frac{1-x}{1+x}\,dx = \int -1 + \frac{2}{x+1}\,dx = \boxed{-x + 2\ln|x+1| + C}$

$$\begin{array}{r} -1 \\ x+1\,\overline{)\,-x+1} \\ \underline{-x-1} \\ 2 \end{array}$$

Ex $\displaystyle\int \frac{1-\cos(x)}{1+\cos(x)}\,dx = \int -1 + \frac{2}{1+\cos(x)}\,dx = -x + 2\int \frac{1}{1+\cos(x)}\,dx$

$\displaystyle = -x + 2\int \frac{1}{1+\cos(x)}\,\frac{1-\cos(x)}{1-\cos(x)}\,dx = -x + 2\int \frac{1-\cos(x)}{1-\cos^2(x)}\,dx$

$\displaystyle = -x + 2\int \frac{1-\cos(x)}{\sin^2(x)}\,dx = -x + 2\int \frac{1}{\sin^2(x)} - \frac{\cos(x)}{(\sin(x))^2}\,dx$

$\displaystyle = -x + 2\int \csc^2(x)\,dx - 2\int (\sin(x))^{-2}\cos(x)\,dx$

$\displaystyle = -x - 2\cot(x) - 2\,\frac{\sin(x)^{-2+1}}{-2+1} + C$

$\displaystyle = \boxed{-x - 2\cot(x) + \frac{2}{\sin(x)} + C}$

Ex $$\int_{\frac{1}{e}}^{1} \frac{dx}{x(\ln^2 x + 2\ln x + 2)} = \int_{\ln(\frac{1}{e})}^{\ln(1)} \frac{1}{u^2+2u+2}\, du$$

$u = \ln x$
$du = \frac{1}{x}\, dx$

$$= \int_{-1}^{0} \frac{du}{u^2+2u+1+1} = \int_{-1}^{0} \frac{du}{(u+1)^2+1}$$

$z = u+1$
$dz = du$

$$= \int_{-1+1}^{0+1} \frac{dz}{z^2+1} = \left[\tan^{-1}(z)\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \boxed{\frac{\pi}{4}}$$

Ex $$\int \frac{\tan\theta + \tan^3\theta}{(1+\tan\theta)^{50}}\, d\theta = \int \frac{\tan\theta\,(1+\tan^2\theta)}{(1+\tan\theta)^{50}}\, d\theta$$

$$= \int \frac{\tan\theta \sec^2(\theta)\, d\theta}{(1+\tan\theta)^{50}} = \int \frac{u-1}{u^{50}}\, du = \int\left(\frac{u}{u^{50}} - \frac{1}{u^{50}}\right) du$$

$u = 1+\tan(\theta)$
$du = \sec^2(\theta)\, d\theta$
$\tan\theta = u-1$

$$= \int u^{-49}\, du - \int u^{-50}\, du$$

$$= \frac{u^{-48}}{-48} - \frac{u^{-49}}{-49} + C$$

$$= \boxed{\frac{1}{49(1+\tan\theta)^{49}} - \frac{1}{48(1+\tan\theta)^{48}} + C}$$

Ex $$\int \sin(x) \ln(\sin(x))\, dx = -\cos(x)\ln(\sin(x)) - \int -\cos(x)\frac{\cos(x)}{\sin(x)}\, dx$$

$u = \ln(\sin(x))$ $\quad dv = \sin(x)\, dx$
$du = \frac{\cos(x)}{\sin(x)}\, dx$ $\quad v = -\cos(x)$

$$= -\cos(x)\ln(\sin(x)) + \int \frac{\cos^2(x)}{\sin(x)}\, dx$$

$$= -\cos(x)\ln(\sin(x)) + \int \frac{1-\sin^2(x)}{\sin(x)}\, dx$$

$$= -\cos(x)\ln(\sin(x)) + \int \frac{1}{\sin(x)} - \sin(x)\, dx$$

$$= -\cos(x)\ln(\sin(x)) + \int \csc(x)\, dx - \int \sin(x)\, dx$$

$$= \boxed{-\cos(x)\ln(\sin(x)) - \ln|\csc(x) + \cot(x)| + \cos(x) + C}$$