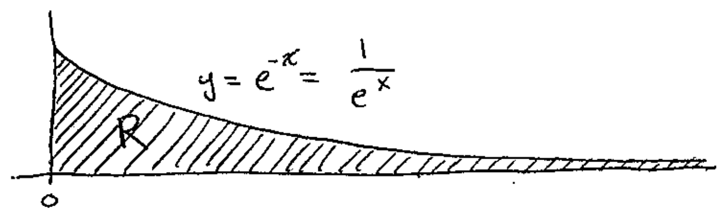
Section 8.9 Improper Integrals

<u>Definition</u> An integral of one of the forms $\int_a^{+\infty} f(x)\,dx$, $\int_{-\infty}^{a} f(x)\,dx$, or $\int_{-\infty}^{\infty} f(x)\,dx$ is called an <u>improper</u> <u>integral</u>.

<u>Today's goal</u> Learn how to compute improper integrals.

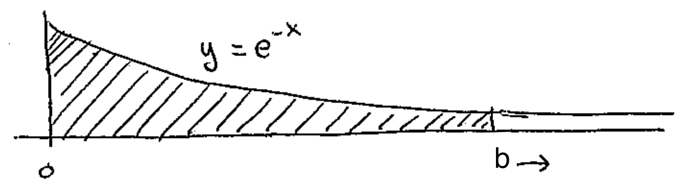
<u>Motivational Question</u>: What is the area of the shaded region? We want to say $\int_0^\infty e^{-x}\,dx$, but how do we compute this improper integral?



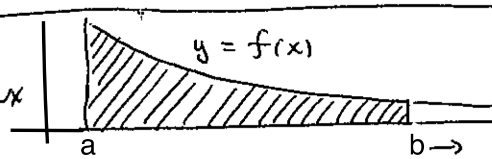
<u>Idea</u>: Area $= \lim\limits_{b\to\infty} \int_0^b e^{-x}\,dx$

$$= \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b = \lim_{b\to\infty}\Big[-e^{-b}+e^{-0}\Big]$$

$$= \lim_{b\to\infty}\Big[-\frac{1}{e^b}+1\Big] = 0+1 = 1 \text{ square unit.}$$



<u>Definition</u> $\int_a^{+\infty} f(x)\,dx = \lim\limits_{b\to+\infty} \int_a^b f(x)\,dx$



If the limit exists we say the integral <u>converges</u>, otherwise it <u>diverges</u>.

$$\int_1^{+\infty} \frac{1}{x^3}\,dx = \lim_{b\to+\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty}\Big[\frac{-1}{2x^2}\Big]_1^b = \lim_{b\to\infty}\Big[\frac{-1}{2b^2}-\frac{-1}{2\cdot 1}\Big] = 0+\frac{1}{2} = \frac{1}{2}$$

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\Big[\ln x\Big]_1^b = \lim_{b\to\infty}\Big[\ln b - \ln 1\Big] = \lim_{b\to\infty} \ln b \leftarrow \text{diverges.}$$

<u>Theorem</u> $$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1} & \text{if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$$ ← probably best not to remember this Theorem ... just work it out each time

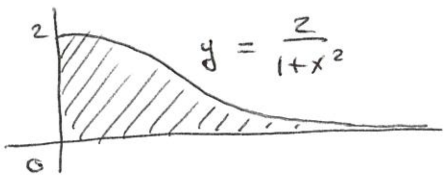
$$\int_1^\infty \Big(\frac{1}{x^2}+\frac{1}{e^{x-1}}\Big)dx = \int_1^\infty (x^{-2}+e^{-x+1})\,dx = \lim_{b\to\infty}\int_1^b (x^{-2}+e^{-x+1})\,dx$$

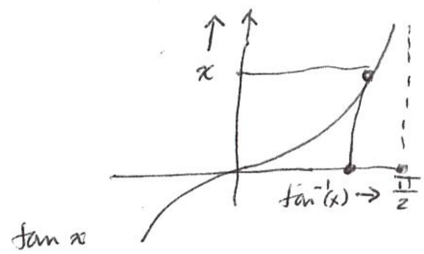
$$= \lim_{b\to\infty}\Big[-x^{-1}-e^{-x+1}\Big]_1^b = \lim_{b\to\infty}\Big[-\frac{1}{x}-\frac{1}{e^{x-1}}\Big]_1^b = \lim_{b\to\infty}\Big[\Big(-\frac{1}{b}-\frac{1}{e^{b-1}}\Big)-\Big(-\frac{1}{1}-\frac{1}{e^0}\Big)\Big]$$

$$= \lim_{b\to\infty}\Big[2-\frac{1}{b}-\frac{1}{e^{b-1}}\Big] = 2-0-0 = 2$$

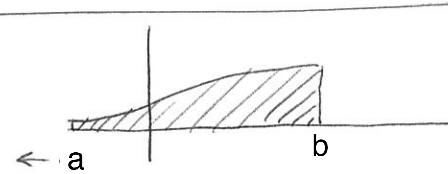
Ex $\displaystyle\int_0^\infty \frac{2}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{2\,dx}{1+x^2} = 2\lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = 2\lim_{b\to\infty}\Big[\tan^{-1}x\Big]_0^b$

$\displaystyle = 2\lim_{b\to\infty}\left[\tan^{-1}(b) - \tan^{-1}(0)\right] = 2\lim_{b\to\infty}\tan^{-1}(b) = 2\cdot\frac{\pi}{2} = \pi$



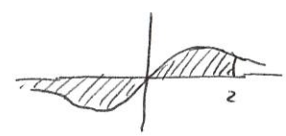


Definition $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$

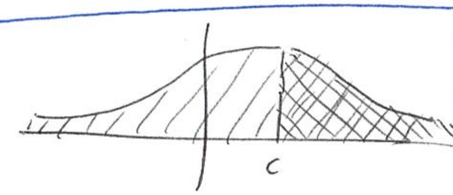


Ex $\displaystyle\int_{-\infty}^{2} \frac{x}{(1+x^2)^2}\,dx = \lim_{a\to-\infty}\int_a^2 \frac{x}{(1+x^2)^2}\,dx = \lim_{a\to-\infty}\frac{-1}{2(1+x^2)}\Big]_a^2$

$\displaystyle = \lim_{a\to-\infty}\left[\frac{-1}{2(1+2^2)} - \frac{-1}{2(1+a^2)}\right] = -\frac{1}{10}$



Definition $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$



Ex $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\,dx = \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx + \int_{0}^{\infty} \frac{x}{(1+x^2)^2}\,dx$

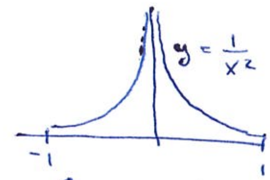
$\displaystyle = \lim_{a\to-\infty}\int_a^0 \frac{x}{(1+x^2)^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{x}{(1+x^2)^2}\,dx$

$\displaystyle = \lim_{a\to-\infty}\left[\frac{-1}{2(1+x^2)}\right]_a^0 + \lim_{b\to\infty}\left[\frac{-1}{2(1+x^2)}\right]_0^b = \lim_{a\to\infty}\left(-\frac{1}{2} + \frac{1}{2(1+a^2)}\right) + \lim_{b\to\infty}\left(-\frac{1}{2(1+b^2)} + \frac{1}{2}\right)$

$\displaystyle = -\frac{1}{2} + 0 + 0 + \frac{1}{2} = 0$

Integrals involving infinite discontinuities

Consider the following (incorrect) computation:



$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{1} x^{-2}\,dx = \left[\frac{-1}{x}\right]_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2$$

This can't be right because $\frac{1}{x^2} > 0$ on $[-1,1]$ so the integral must be positive. The problem is that $\frac{1}{x^2}$ has an infinite discontinuity at $x=0$, so the F.T.C. is not guaranteed to work:

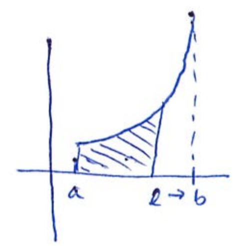
> F.T.C.: If f is <u>continuous</u> on $[a,b]$, and $F(x)$ is any antiderivative of f, then $\int_a^b f(x)\,dx = F(x)\Big]_a^b$

An integral such as $\int_{-1}^{1} \frac{1}{x^2}\,dx$ which has one or more ∞ discontinuities on $[a,b]$ is also called an <u>improper integral</u>. Now we'll talk about how to deal with these.

Definitions

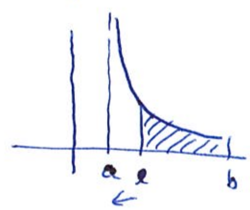
If there is an ∞ discontinuity at b then

$$\int_a^b f(x)\,dx = \lim_{l \to b^-} \int_a^l f(x)\,dx$$



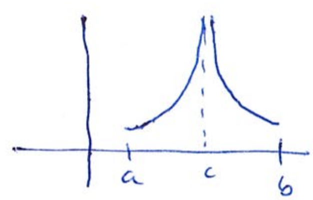
If there is an ∞ discontinuity at a then

$$\int_a^b f(x)\,dx = \lim_{l \to a^+} \int_l^b f(x)\,dx$$

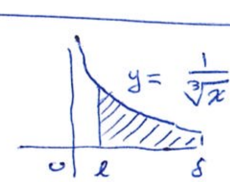


If there is an ∞ discontinuity at c in $[a,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



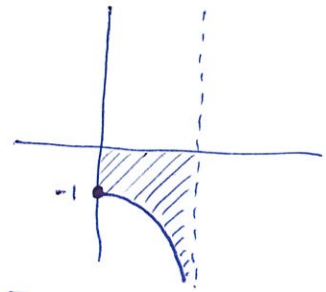
Ex $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx = \lim_{l \to 0^+} \int_l^8 x^{-\frac{1}{3}}\,dx = \lim_{l \to 0^+} \left[\frac{3}{2} x^{\frac{2}{3}}\right]_l^8$



$$\lim_{l \to 0^+} \left[\frac{3\sqrt[3]{x}^{\,2}}{2}\right]_l^8 = \lim_{l \to 0^+} \left[\frac{3\sqrt[3]{8}^{\,2}}{2} - \frac{3\sqrt[3]{l}^{\,2}}{2}\right] = 6$$

Ex

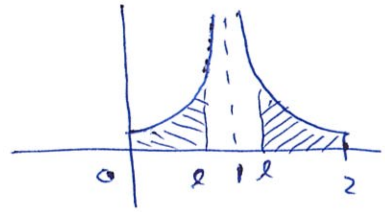
$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{l\to 1^-} \int_0^l \frac{1}{x-1}\,dx$$



$$= \lim_{l\to 1^-} \ln|x-1| \Big]_0^l = \lim_{l\to 1^-} \Big[\ln|l-1| - \ln|0-1|\Big]$$

$$= \lim_{l\to 1^-} \Big(\ln|l-1| \Big) = \infty$$ Integral diverges.

Ex



$$\int_0^2 \frac{dx}{(1-x)^{2/3}} = \int_0^2 (1-x)^{-\frac{2}{3}}\,dx$$

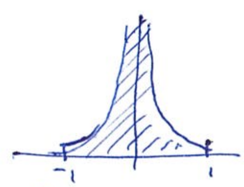
$$= \int_0^1 (1-x)^{-\frac{2}{3}}\,dx + \int_1^2 (1-x)^{-\frac{2}{3}}\,dx$$

$$= \lim_{l\to 1^-} \int_0^l (1-x)^{-2/3}\,dx + \lim_{l\to 1^+} \int_l^2 (1-x)^{-2/3}\,dx$$

$$= \lim_{l\to 1^-} \Big[-3\sqrt[3]{1-x}\Big]_0^l + \lim_{l\to 1^+} \Big[-3\sqrt[3]{1-x}\Big]_l^2$$

$$= \lim_{l\to 1^-} \Big[-3\sqrt[3]{1-l} + 3\sqrt{1-0}\Big] + \lim_{l\to 1^+} \Big[-3\sqrt[3]{1-2} + 3\sqrt[3]{1-l}\Big]$$

$$= (0+3) + (-(-3)+0) = 6$$



Ex $$\int_{-1}^1 \frac{1}{x^2}\,dx = \int_{-1}^0 x^{-2}\,dx + \int_0^1 x^{-2}\,dx$$

$$= \lim_{l\to 0^-} \int_{-1}^l x^{-2}\,dx + \lim_{l\to 0^+} \int_l^1 x^{-2}\,dx = \lim_{l\to 0^-} \Big[\frac{-1}{x}\Big]_{-1}^l + \lim_{l\to 0^+} \Big[\frac{-1}{x}\Big]_l^1$$

$$= \lim_{l\to 0^-} \Big[\frac{-1}{l} + 1\Big] + \lim_{l\to 0^+} \Big[\frac{-1}{1} - \frac{-1}{l}\Big] = \infty + 1 - 1 + \infty = \infty$$ Diverges

Comparison Test

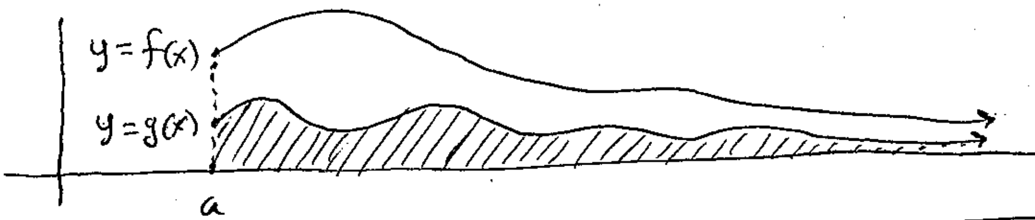
If an improper integral is difficult, this test can at least tell you whether the integral converges or diverges.

Theorem 8.2 Comparison Test

Suppose $0 \le g(x) \le f(x)$ on $[a, \infty)$

① If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

② If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.



Ex Does $\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$ converge or diverge?

$$\underbrace{\frac{\sqrt{x^4+1}}{x^3}}_{f(x)} \ge \frac{\sqrt{x^4}}{x^3} = \frac{x^2}{x^3} = \underbrace{\frac{1}{x}}_{g(x)}$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \Big[\ln x\Big]_1^\infty = \lim_{b\to\infty}\big(\ln(b) - \ln(1)\big) = \infty$$

Therefore $\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$ diverges

OK to write this as $\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx \ge \int_1^\infty \frac{\sqrt{x^4}}{x^3}\,dx = \int_1^\infty \frac{1}{x}\,dx = \infty$

Ex Does $\int_0^\infty \frac{1}{\sin^2(x)+e^x+2}\,dx$ converge or diverge?

$$\int_0^\infty \frac{1}{\sin^2(x)+e^x+2}\,dx \le \int_0^\infty \frac{1}{e^x}\,dx = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b$$

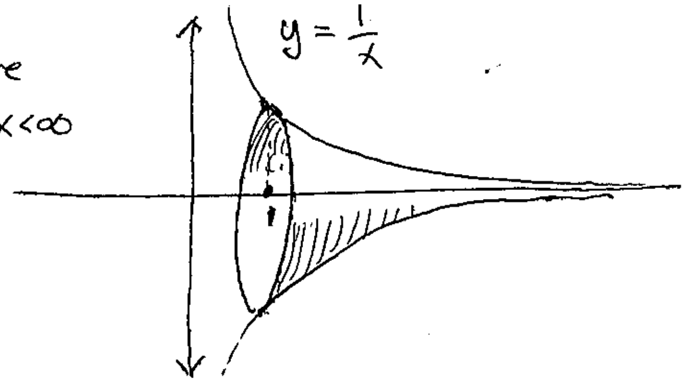
$$= \lim_{b\to\infty}\left(-e^{-b} - (-e^{-0})\right) = 0 + 1 = 1$$

$\boxed{\text{It converges!}}$

Example "Gabriel's Horn"

Here is a shape that has finite volume but infinite surface area. You can fill it with paint, but you can't paint it.

Take the curve $y = \frac{1}{x}$ for $1 \le x < \infty$ and rotate it around the x-axis.

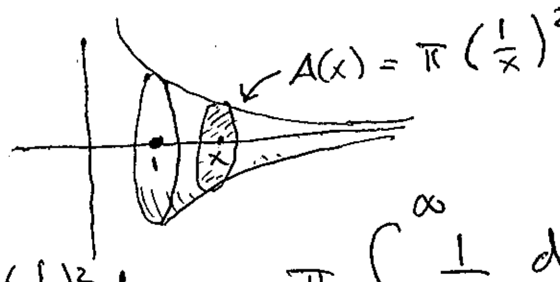


$$SA = \int_1^\infty 2\pi \frac{1}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx = 2\pi \int_1^\infty \frac{1}{x} \sqrt{\frac{x^4+1}{x^4}}\, dx$$

$$= 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\, dx \ge 2\pi \int_1^\infty \frac{\sqrt{x^4}}{x^3}\, dx = 2\pi \int_1^\infty \frac{1}{x}\, dx$$

$$= 2\pi \lim_{b\to\infty} \Big[\ln|x|\Big]_1^b = 2\pi \lim_{b\to\infty} \Big[\ln b - \ln 1\Big] = \infty$$

Therefore the surface area is infinite.



Volume

$$V = \int_1^\infty A(x)\, dx = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \pi \int_1^\infty \frac{1}{x^2}\, dx$$

$$= \pi \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \pi \lim_{b\to\infty} \left(-\frac{1}{b} - \left(-\frac{1}{1}\right)\right) = \pi$$

Therefore the volume is π cubic units (finite!)