Name: Richard

FINAL EXAM ♣

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MATH 201

R. Hammack

1. $\int x^2 e^x \, dx =$ $x^2e^x - 2\int xe^x dx = x^2e^x - 2\left(xe^x - \int e^x dx\right)$

$\boxed{\begin{matrix} u = x^2 & dv = e^x dx \\ du = 2x\,dx & v = e^x \end{matrix}}$ $\boxed{\begin{matrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{matrix}}$

$= x^2e^x - 2xe^x + 2e^x + C$

$= \boxed{e^x(x^2 - 2x + 2) + C}$

2. $\int \frac{(1+\ln(x))^5 \ln(x)}{x} \, dx =$ $\int u^5(u-1)\, du = \int u^6 - u^5 \, du$

$\boxed{\begin{matrix} u = 1 + \ln(x) \\ du = \frac{1}{x} dx \\ \ln(x) = u - 1 \end{matrix}}$

$= \frac{u^7}{7} - \frac{u^6}{6} + C$

$= \boxed{\frac{(1+\ln(x))^7}{7} - \frac{(1+\ln(x))^6}{6} + C}$

3. $\int \sec^4(x) \tan(x) \, dx =$ $\int \sec^2(x) \tan(x) \sec^2(x) \, dx$

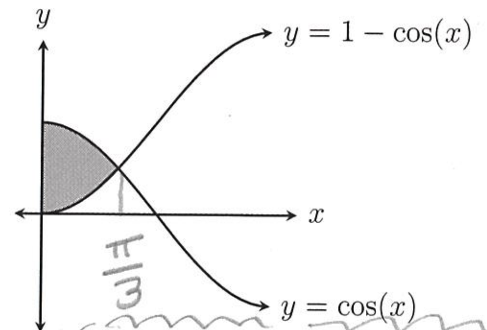
$\boxed{\begin{matrix} u = \tan(x) \\ du = \sec^2(x) dx \end{matrix}}$

$= \int (1 + \tan^2(x)) \tan(x) \sec^2(x) \, dx$

$= \int (1 + u^2) u \, du = \int u + u^3 \, du$

$= \frac{u^2}{2} + \frac{u^4}{4} + C = \boxed{\frac{\tan^2(x)}{2} + \frac{\tan^4(x)}{4} + C}$

4. Find the area of the shaded region.



$$\int_0^{\frac{\pi}{3}} \cos x - (1 - \cos x)\, dx$$

$$= \int_0^{\pi/3} 2\cos x - 1 \, dx$$

$$= \Big[2\sin x - x\Big]_0^{\pi/3} = \left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) - (2\sin 0 - 0)$$

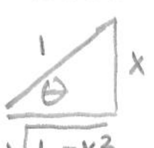
$$= 2\frac{\sqrt{3}}{2} - \frac{\pi}{3} = \boxed{\frac{3\sqrt{3} - \pi}{3} \text{ sq. units}}$$

$1 - \cos(x) = \cos(x)$
$1 = 2\cos(x)$
$\cos(x) = \frac{1}{2}$
$x = \frac{\pi}{3}$

5. $\int \sqrt{1 - x^2}\, dx = \int \sqrt{1 - \sin^2\theta}\, \cos\theta \, d\theta = \int \cos^2\theta \, d\theta = \frac{\theta}{2} + \frac{\cos(\theta)\sin(\theta)}{2}$

$$= \boxed{\frac{\sin^{-1}(x)}{2} + \frac{x\sqrt{1 - x^2}}{2} + C}$$

$x = \sin(\theta)$
$dx = \cos(\theta)\, d\theta$



6. $\int \frac{5 - x}{x^2 - 5x + 6}\, dx = \int \frac{-3}{x-2} + \frac{2}{x-3}\, dx = \boxed{2\ln|x-3| - 3\ln|x-2| + C}$

$$= \boxed{\ln\left|\frac{(x-3)^2}{(x-2)^3}\right| + C}$$

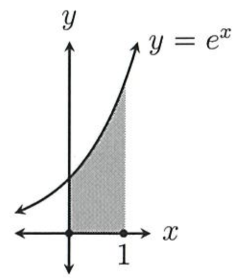
$$\frac{5 - x}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$5 - x = A(x-3) + B(x-2)$$

$x = 2$ $\quad 3 = -A \quad = \boxed{A = -3}$

$x=3$ $\quad$ -2 = B

7. The shaded region is rotated around the x-axis. Find the volume of the resulting solid.



Slicing

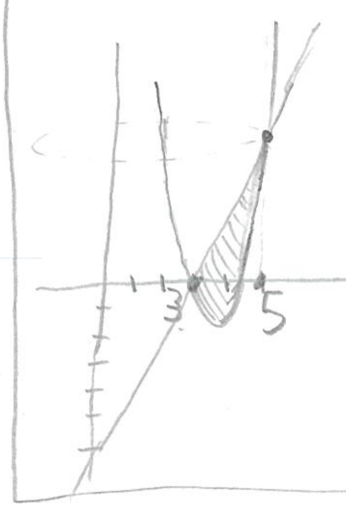
$$\int_0^1 \pi (e^x)^2 \, dx = \pi \int_0^1 e^{2x} \, dx$$

$$= \pi \left[\frac{1}{2} e^{2x} \right]_0^1 = \pi \left(\frac{1}{2} e^2 - \frac{1}{2} e^0 \right)$$

$$= \boxed{\frac{\pi}{2}(e^2 - 1) \text{ cubic units}}$$

8. The region bounded by $f(x) = (x-3)^2$ and $g(x) = 2x - 6$ is rotated around the y-axis. Find the volume of the resulting solid.

$(x-3)^2 = 2x - 6$
$x^2 - 6x + 9 = 2x - 6$
$x^2 - 8x + 15 = 0$
$(x-3)(x-5) = 0$
$x = 3 \quad x = 5$



Shells

$$V = \int_3^5 2\pi x \left((2x-6) - (x-3)^2 \right) dx$$

$$= 2\pi \int_3^5 x(2x - 6 - x^2 + 6x - 9) \, dx = 2\pi \int_3^5 -x^3 + 8x^2 - 15x \, dx$$

$$= 2\pi \left[-\frac{x^4}{4} + \frac{8x^3}{3} - \frac{15x^2}{2} \right]_3^5$$

$$= 2\pi \left(\left(-\frac{5^4}{4} + \frac{8 \cdot 5^3}{3} - \frac{15 \cdot 5^2}{2} \right) - \left(-\frac{3^4}{4} + \frac{8 \cdot 3^3}{3} - \frac{15 \cdot 3^2}{2} \right) \right)$$

$$= 2\pi \left(-\frac{625}{4} + \frac{8 \cdot 125}{3} - \frac{15 \cdot 25}{2} + \frac{81}{4} - \frac{216}{3} + \frac{135}{2} \right)$$

$$= 2\pi \left(-\frac{544}{4} + \frac{784}{3} - \frac{240}{2} \right) = 2\pi \left(-136 + \frac{784}{3} - 120 \right)$$

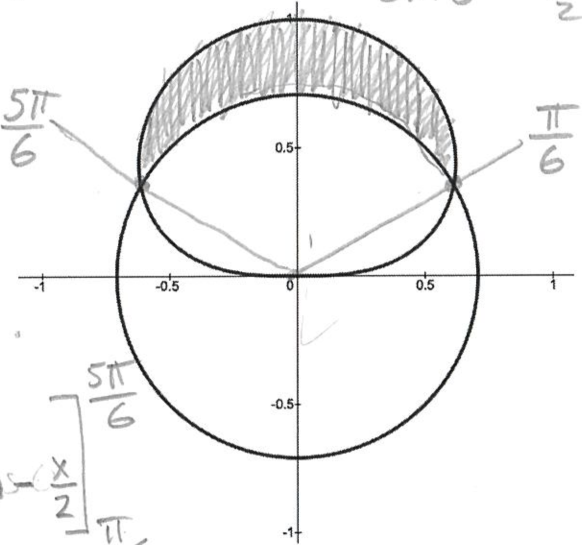
$$= 2\pi \left(-256 + \frac{785}{3} \right) = 2\pi \left(\frac{-768}{3} + \frac{785}{3} \right) = 2\pi \left(\frac{17}{3} \right) = \boxed{\frac{34\pi}{3} \text{ cubic units}}$$

9. The graphs of the polar equations $r = \sqrt{\sin(\theta)}$ and $r = \frac{1}{\sqrt{2}}$ are shown below.

$\sqrt{\sin\theta} = \frac{1}{\sqrt{2}}$

$\sin\theta = \frac{1}{2}$

Find the area inside $r = \sqrt{\sin(\theta)}$ and outside $r = \frac{1}{\sqrt{2}}$.



$$A = \frac{1}{2}\int_{\pi/6}^{\frac{5\pi}{6}} \sqrt{\sin\theta}^{\,2} - \left(\frac{1}{\sqrt{2}}\right)^2 d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \sin\theta - \frac{1}{2}\, d\theta = \frac{1}{2}\left[-\cos\theta - \frac{x}{2}\right]_{\pi/6}^{\frac{5\pi}{6}}$$

$$= \frac{1}{2}\left(-\cos\frac{5\pi}{6} - \frac{5\pi}{12} - \left(-\cos\frac{\pi}{6} - \frac{\pi}{12}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{5\pi}{12} + \frac{\sqrt{3}}{2} + \frac{\pi}{12}\right) = \boxed{\frac{\sqrt{3}}{2} - \frac{\pi}{6} \text{ sq. units}}$$

10. Find the arc length of the curve $y = \ln(x) - \frac{x^2}{8}$ between $x = 1$ and $x = 2$.

$$L = \int_1^2 \sqrt{1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{1}{x}\right)^2 - \frac{1}{2} + \left(\frac{x}{4}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{\left(\frac{1}{x}\right)^2 + \frac{1}{2} + \left(\frac{x}{4}\right)^2}\, dx = \int_1^2 \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\, dx$$

$$= \int_1^2 \frac{1}{x} + \frac{x}{4}\, dx = \left[\ln(x) + \frac{x^2}{8}\right]_1^2 = \left(\ln(2) + \frac{1}{2} - \ln(1) - \frac{1}{8}\right)$$

$$= \boxed{\ln(2) + \frac{3}{8} \text{ units}}$$

11. $\int x\, e^{x/3}\, dx =$ $3xe^{x/3} - \int 3e^{x/3}\, dx$

$u = x \quad dv = e^{x/3}\, dx$
$du = dx \quad v = 3e^{x/3}$

$= \boxed{3xe^{x/3} - 9e^{x/3} + C}$

12. Find $\int_{-\infty}^{0} x\, e^{x/3} dx.$ $= \lim_{b \to -\infty} \int_b^0 x e^{x/3}\, dx$

$= \lim_{b \to -\infty} \left[3xe^{x/3} - 9e^{x/3} \right]_b^0$

$= \lim_{b \to -\infty} \left((3 \cdot 0 e^0 - 9e^0) - (3be^{b/3} - 9e^{b/3}) \right)$

$= \lim_{b \to -\infty} \left(0 - 9 - 3be^{b/3} - 9e^{b/3} \right)$

$= -9 - \lim_{b \to -\infty} \frac{-3b}{e^{-b/3}} - \lim_{b \to -\infty} \frac{9}{e^{-b/3}}$

$= -9 - \lim_{b \to -\infty} \frac{-3}{-\frac{1}{3}e^{-b/3}} - 0 = -9 - 0 = \boxed{-9}$

13. Does the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots$ converge? If so, to what number?

Converges by AST,

converges to $\ln(2)$ because

$\ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

14. What function is represented by the power series $\sum_{k=0}^{\infty} x^{2k}$? $= 1 + x^2 + x^4 + x^6 + x^8 + \ldots$

$x^2 \quad x^2 \quad x^2$

$$\boxed{f(x) = \frac{1}{1-x^2}}$$

geometric series

$a = 1$

$r = x^2$

converges to $\frac{1}{1-x^2}$ if $|x|<1$

15. Find the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k+1}$. Be sure to test endpoints, if appropriate.

$$\lim_{k\to\infty} \left| \frac{\frac{x^{k+1}}{k+2}}{\frac{x^k}{k+1}} \right| = \lim_{k\to\infty} \left| \frac{k+1}{k+2} x \right| = |x| < 1$$

Interval is $-1 \le x \le 1$

Test $x=1$ $\quad \sum_{k=1}^{\infty} \frac{(-1)^k 1^k}{k+1} = \frac{-1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots$ converges (alternating harmonic)

Test $x=-1$ $\quad \sum_{k=1}^{\infty} \frac{(-1)^k(-1)^k}{k+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \cdots$ diverges (harmonic)

Interval of convergence is $\boxed{(-1, 1]}$

16. Use any appropriate test to determine if the series $\sum_{k=1}^{\infty} \frac{\ln(k)}{\ln(k+1)}$ converges or diverges.

$$\lim_{k\to\infty} \frac{\ln(k)}{\ln(k+1)} = \lim_{k\to\infty} \frac{1/k}{1/(k+1)} = \lim_{k\to\infty} \frac{k+1}{k} = 1$$

Diverges by divergence test.

17. Use any appropriate test to determine if the series $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k-1}}$ converges or diverges.

$$\frac{1}{k^{1/2}} = \frac{1}{\sqrt{k}} < \frac{1}{\sqrt{k-1}}$$

Diverges by comparison with divergent p-series $\sum \frac{1}{k^{1/2}}$

18. Use the Maclaurin series for e^x to obtain a power series representation for $g(x) = \frac{e^x - 1 - x}{x}$.

$$g(x) = \frac{1}{x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots - 1 - x\right)$$

$$= \frac{1}{x}\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

$$= \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \cdots = \boxed{\sum_{k=1}^{\infty} \frac{x^k}{(k+1)!}}$$

19. Use the Binomial Theorem to write the first three terms of a power series for the function $(1+x)^{1/2}$.

$$1 + \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \cdots$$

$$= \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots}$$

20. Write a third-degree Taylor polynomial $p_3(x)$ centered at $x = 2$ for the function $f(x) = \ln(3x - 5)$.

$f^0(x) = \ln(3x-5) \qquad f^{(0)}(2) = \ln(1) = 0$

$f'(x) = \dfrac{3}{3x-5} \qquad f^{(1)}(2) = 3$

$f''(x) = \dfrac{-9}{(3x-5)^2} \qquad f^{(2)}(2) = -9$

$f'''(x) = \dfrac{54}{(3x-5)^2} \qquad f^{(3)}(2) = 54$

$$p_3(x) = \ln(1) + 3(x-2) - \frac{9(x-2)^2}{2!} + \frac{54(x-2)^3}{3!}$$

$$= \boxed{3(x-2) - \frac{9}{2}(x-2)^2 + 9(x-2)^3}$$