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Test 1
February 13, 2025

MATH 201
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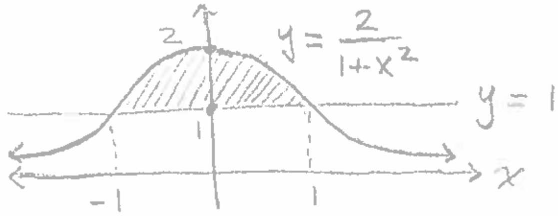
1. Find the area of the region contained between the graphs of $y = \dfrac{2}{1+x^2}$ and $y = 1$.



Find intersections

$\dfrac{2}{1+x^2} = 1 \Rightarrow 2 = 1 + x^2$

$\Rightarrow 1 = x^2$

$\Rightarrow x = \pm 1$

$$A = \int_{-1}^{1} \left(\frac{2}{1+x^2} - 1 \right) dx = \Big[2\tan^{-1}(x) - x \Big]_{-1}^{1}$$

$$= \left(2\tan^{-1}(1) - 1 \right) - \left(2\tan^{-1}(-1) - (-1) \right)$$

$$= \left(2\frac{\pi}{4} - 1 \right) - \left(2\left(-\frac{\pi}{4}\right) + 1 \right) = \boxed{\pi - 2 \text{ square units}}$$

2. Consider the region contained between the graphs of $y = x^3 - 2x^2 + x$ and $y = 0$.
This region is revolved around the y-axis. Find the volume of the resulting solid.

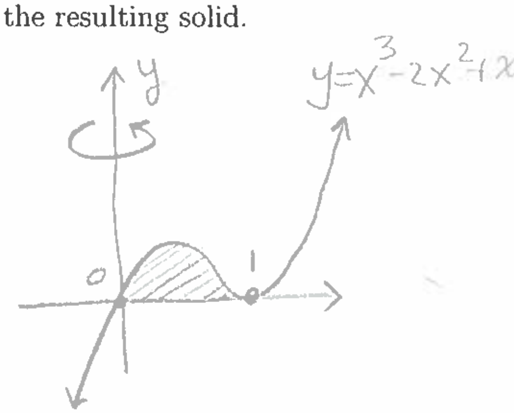
$y = x^3 - 2x^2 + x$

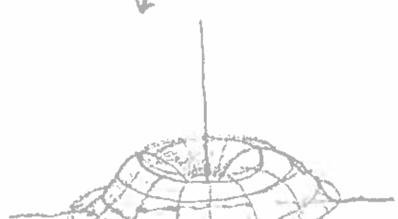
$= x(x^2 - 2x + 1)$

$= x(x-1)^2$

(x-intercepts)

$x = 0 \qquad x = 1$





Volume by shells

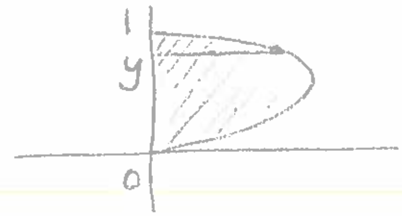
$$V = \int_0^1 2\pi x f(x)\, dx$$

$$2\pi \int_0^1 x(x^3 - 2x^2 + x)\, dx = 2\pi \int_0^1 x^4 - 2x^3 + x^2 \, dx$$

$$= 2\pi \left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} \right]_0^1 = 2\pi \left(\frac{1}{5} - \frac{1}{2} + \frac{1}{3} \right) = 2\pi \left(\frac{6}{30} - \frac{15}{30} + \frac{10}{30} \right)$$

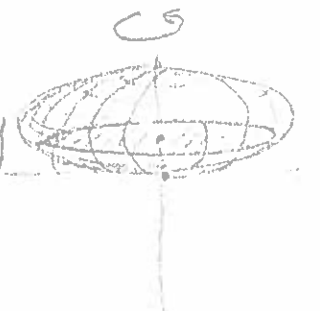
$$= \boxed{\frac{\pi}{15} \text{ cubic units}}$$

3. Consider the region contained between the y-axis and the curve $x = y - y^2$. $= y(y-1)$
This region is revolved around the y-axis. What is the volume of the resulting solid?



Cross-sectional area at y is $A(y) = \pi(y-y^2)^2 =$

$\pi(y^2 - 2y^3 + y^4)$.



$$V = \int_0^1 A(y)\,dy = \pi\int_0^1 (y^2 - 2y^3 + y^4)\,dy$$

$$= \pi\left[\frac{y^3}{3} - 2\frac{y^4}{4} + \frac{y^5}{5}\right]_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \pi\left(\frac{10}{30} - \frac{15}{30} + \frac{6}{30}\right) = \boxed{\frac{\pi}{30} \text{ cubic units}}$$

4. Find the area of the surface obtained by rotating $y = 2\sqrt{x}$ for $0 \le x \le 3$ around the x-axis.

$$A = \int_0^3 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx = 4\pi\int_0^3 \sqrt{x}\sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\,dx$$

$$= 4\pi\int_0^3 \sqrt{x\left(1 + \frac{1}{x}\right)}\,dx = 4\pi\int_0^3 \sqrt{x+1}\,dx$$

$$= 4\pi\left[\frac{2\sqrt{x+1}^3}{3}\right]_0^3 = 4\pi\left(\frac{2\sqrt{3+1}^3}{3} - \frac{2\sqrt{0+1}^3}{3}\right)$$

$$= 4\pi\left(\frac{16}{3} - \frac{2}{3}\right) = \boxed{\frac{56}{3}\pi \text{ square units}}$$

5. Find the arc length of the curve $y = \int_0^x \sqrt{t^2+2t}\,dt$ from $x=2$ to $x=4$.

By the Fundamental Theorem of Calculus,

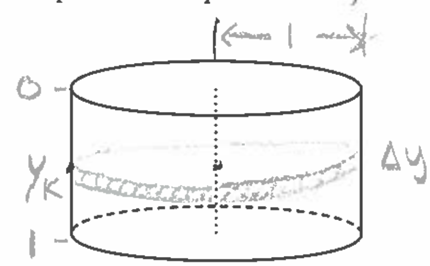
$y' = \sqrt{x^2+2x}$.

Now, $L = \int_2^4 \sqrt{1+(y')^2}\,dx = \int_2^4 \sqrt{1+\sqrt{x^2+2x}^{\,2}}\,dx$

$= \int_2^4 \sqrt{x^2+2x+1}\,dx = \int_2^4 \sqrt{(x+1)^2}\,dx = \int_2^4 x+1\,dx$

$= \left[\frac{x^2}{2}+x\right]_2^4 = \left(\frac{4^2}{2}+4\right)-\left(\frac{2^2}{2}+2\right) = 12-4 = \boxed{8 \text{ units}}$

6. A cylindrical tank, filled with water, is 1 meter high, and has a radius of 1 meter. Calculate the work required to pump all the water to the top of the tank. (Recall that the density of water is 1000 kilograms per cubic meter, and the acceleration due to gravity is 9.8 meters per second per second.)



Layer #k has a volume of $\pi \cdot 1^2 \Delta y = \pi \Delta y$ cubic meters.

Its mass is thus $1000\pi\Delta y$ kg.

It must be pumped a distance of y_k meters,

Work done to pump layer #k is therefore

$mad = 1000\pi\Delta y \cdot 9.8\, y_k = 9800\pi\, y_k \Delta y$ J.

Total work done is $\lim_{n\to\infty} \sum_{k=1}^{n} 9800\pi\, y_k \Delta y$

$= \int_0^1 9800\pi y\,dy = 9800\pi\left[\frac{y^2}{2}\right]_0^1 = \boxed{4900\pi \text{ J}}$