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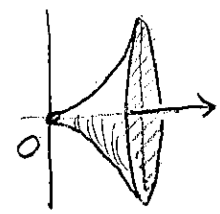
TEST 2 — March 27, 2025

MATH 201 — R. Hammack

1. The region under $y = \tan(x)$ and over $\left[0, \frac{\pi}{4}\right]$ is rotated around the x-axis. Find the volume.

Volume by slicing:

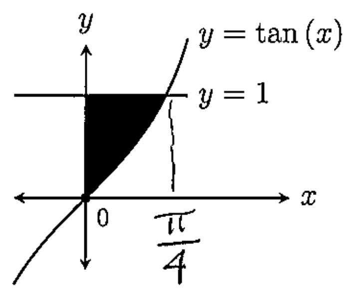


$$V = \int_0^{\pi/4} \pi\left(\tan(x)\right)^2 dx = \pi \int_0^{\pi/4} \tan^2(x)\,dx$$

$$= \pi\Big[\tan(x) - x\Big]_0^{\pi/4} = \pi\left(\left(\tan\left(\tfrac{\pi}{4}\right) - \tfrac{\pi}{4}\right) - \left(\tan(0) - 0\right)\right)$$

$$= \pi\left(1 - \frac{\pi}{4}\right) = \boxed{\frac{(4-\pi)\pi}{4} \text{ cubic units}}$$

2. Find the area of the shaded region.



$$A = \int_0^{\pi/4} 1 - \tan(x)\,dx$$

$$= \Big[x - \ln|\sec(x)|\Big]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} - \ln\left|\sec\left(\tfrac{\pi}{4}\right)\right|\right) - \left(0 - \ln(\sec(0))\right)$$

$$= \frac{\pi}{4} - \ln(\sqrt{2}) - (0 - \ln(1)) = \boxed{\frac{\pi}{4} - \ln(\sqrt{2}) \text{ square units}}$$

3. $\displaystyle\int \frac{\ln(x)}{x^4}\,dx = \ln(x)\left(\frac{-1}{3x^3}\right) - \int \frac{-1}{3x^3}\,\frac{1}{x}\,dx$

Integration by parts

$u = \ln(x) \qquad dv = x^{-4}dx$

$du = \dfrac{1}{x}dx \qquad v = \dfrac{x^{-3}}{-3} = \dfrac{-1}{3x^3}$

$$= \frac{-\ln(x)}{3x^3} + \int \frac{1}{3x^4}\,dx$$

$$= -\frac{\ln(x)}{3x^3} + \frac{1}{3}\int x^{-4}\,dx$$

$$= \frac{-\ln(x)}{3x^2} + \frac{1}{3}\,\frac{x^{-3}}{-3} = \boxed{\frac{-\ln(x)}{3x^2} - \frac{1}{9x^3} + C}$$

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4. $\displaystyle\int \sec^4(x)\,dx = \int \sec^2(x)\,\sec^2(x)\,dx$

$$= \int (1+\tan^2(x))\,\sec^2(x)\,dx \quad \leftarrow \quad u = \tan(x),\ du = \sec^2(x)\,dx$$

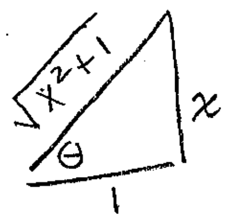
$$= \int 1 + u^2\,du = u + \frac{u^3}{3} + C$$

$$= \boxed{\tan(x) + \frac{\tan^3(x)}{3} + C}$$

5. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2(\theta)}{\tan^2(\theta)\sqrt{\tan^2(\theta)+1}}\,d\theta$

$x = \tan(\theta)$

$dx = \sec^2(\theta)\,d\theta$



$$= \int \frac{\sec^2(\theta)}{\tan^2(\theta)\sec(\theta)}\,d\theta$$

$$= \int \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta = \int \frac{\frac{1}{\cos(\theta)}}{\frac{\sin^2(\theta)}{\cos^2(\theta)}}\,d\theta$$

$$= \int \frac{\cos(\theta)}{\sin^2(\theta)}\,d\theta = \int \frac{1}{\sin(\theta)}\,\frac{\cos(\theta)}{\sin(\theta)}\,d\theta$$

$$= \int \csc(\theta)\cot(\theta)\,d\theta = -\csc(\theta) + C$$

$$= -\frac{\text{HYP}}{\text{ADJ}} + C = \boxed{\frac{-\sqrt{x^2+1}}{x} + C}$$

6. Use integration by parts to find $\displaystyle\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int x\,\frac{1}{1+x^2}\,dx$

$u = \tan^{-1}(x)$ $\quad dv = dx$

$du = \dfrac{1}{1+x^2}\,dx$ $\quad v = x$

$$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln\left|1+x^2\right| + C}$$

7. $\int \frac{8}{x^2+4x-12}\,dx = \int \frac{8}{(x-2)(x+6)}\,dx = \int \frac{A}{x-2} + \frac{B}{x+6}\,dx$

$\frac{8}{(x-2)(x+6)} = \frac{A}{x-2} + \frac{B}{x+6}$

$8 = A(x+6) + B(x-2)$

$x=2 \Rightarrow 8 = 8A \Rightarrow \boxed{A=1}$

$x=-6 \Rightarrow 8 = -8B \Rightarrow \boxed{B=-1}$

$= \int \frac{1}{x-2} - \frac{1}{x+6}\,dx$

$= \ln|x-2| - \ln|x+6| + C$

$= \boxed{\ln\left|\frac{x-2}{x+6}\right| + C}$

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8. $\int_2^\infty \frac{\sin(\pi/x)}{x^2}\,dx = \lim_{b\to\infty} \int_2^b \sin\left(\frac{\pi}{x}\right)\frac{1}{x^2}\,dx$

$u = \pi/x$

$du = \frac{-\pi}{x^2}\,dx$

$-\frac{1}{\pi}du = \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty} \int_{\frac{\pi}{2}}^{\frac{\pi}{b}} \sin(u)\left(-\frac{1}{\pi}\right)du$

$= -\frac{1}{\pi} \lim_{b\to\infty} \int_{\frac{\pi}{2}}^{\frac{\pi}{b}} \sin(u)\,du$

$= -\frac{1}{\pi} \lim_{b\to\infty} \Big[-\cos(u)\Big]_{\pi/2}^{\pi/b} = -\frac{1}{\pi} \lim_{b\to\infty} \left(-\cos\left(\frac{\pi}{b}\right) + \cos\left(\frac{\pi}{2}\right)\right)$

$= -\frac{1}{\pi}\left(-\cos(0) + 0\right) = \boxed{\frac{1}{\pi}}$

9. $\int_2^3 x(x-2)^9\,dx =$ $\int_{2-2}^{3-2} (u+2)u^9\,du$

$u = x-2$

$du = dx$

$x = u+2$

$= \int_0^1 u^{10} + 2u^9\,du$

$= \left[\frac{u^{11}}{11} + \frac{u^{10}}{5}\right]_0^1$

$= \frac{1}{11} + \frac{1}{5} = \frac{5}{55} + \frac{11}{55} = \boxed{\frac{16}{55}}$

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10. $\int \frac{x^2+2x+4}{x+1}\,dx =$ $\int x+1 + \frac{3}{x+1}\,dx$

$$\begin{array}{r@{\,}l} & x+1 \\ x+1 \,\big) & x^2+2x+4 \\ & \underline{x^2+x} \\ & x+4 \\ & \underline{x+1} \\ & 3 \end{array}$$

$= \boxed{\frac{x^2}{2} + x + 3\ln|x+1| + C}$