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TEST 2
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MATH 201
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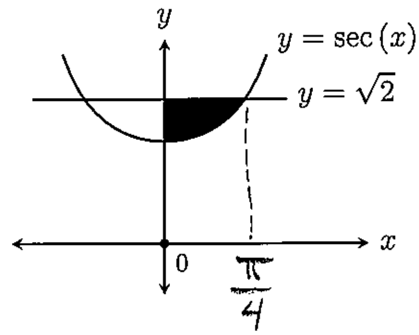
1. Find the area of the shaded region.



$$A = \int_0^{\frac{\pi}{4}} \sqrt{2} - \sec(x)\, dx$$

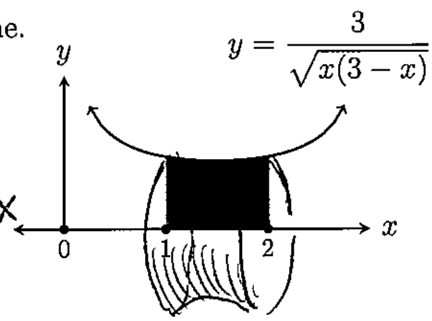
$$= \left[\sqrt{2}\,x - \ln|\sec(x) + \tan(x)|\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\sqrt{2}\pi}{4} - \ln\left|\sec\left(\tfrac{\pi}{4}\right) + \tan\left(\tfrac{\pi}{4}\right)\right|\right) - \left(\sqrt{2}\cdot 0 - \ln|\sec(0) - \tan(0)|\right)$$

$$= \frac{\sqrt{2}\pi}{4} - \ln|\sqrt{2}+1| - (0 - \ln|1-0|)$$

$$= \boxed{\frac{\sqrt{2}\pi}{4} - \ln|\sqrt{2}+1| \text{ square units}}$$

2. The shaded region is rotated around the x-axis. Find the volume.



Volume by slicing:

$$V = \int_1^2 \pi\left(\frac{3}{\sqrt{x(3-x)}}\right)^2 dx = \pi\int_1^2 \frac{9}{x(3-x)}\, dx$$

$$= \pi\int_1^2 \frac{A}{x} + \frac{B}{3-x}\, dx$$

$$\frac{9}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x}$$

$$9 = A(3-x) + Bx$$

$$x=0 \Rightarrow 9 = A\cdot 3 \Rightarrow A=3$$
$$x=3 \Rightarrow 9 = B\cdot 3 \Rightarrow B=3$$

$$= \pi\int_1^2 \frac{3}{x} + \frac{3}{3-x}\, dx$$

$$= \pi\left[3\ln|x| - 3\ln|3-x|\right]_1^2$$

$$= \pi\left[3\ln\left|\frac{x}{3-x}\right|\right]_1^2 = \pi\left(3\ln(2) - 3\ln\left(\tfrac{1}{2}\right)\right)$$

$$= \pi\left(3\ln(2) + 3\ln(2)\right) = \boxed{6\pi\ln(2) \text{ cubic units}}$$

3. $\int \tan^4(x)\,dx = \int \tan^2(x)\tan^2(x)\,dx$

$$= \int \tan^2(x)\left(\sec^2(x) - 1\right)dx$$

$$= \int \tan^2(x)\sec^2(x) - \tan^2(x)\,dx$$

$$= \int (\tan(x))^2 \sec^2(x)\,dx - \int \tan^2(x)\,dx$$

$$= \frac{\tan^3(x)}{3} - (\tan x - x) + C$$

$$= \boxed{\frac{\tan^3(x)}{3} - \tan x + x + C}$$

4. $\int x^5 \ln(x)\,dx = \ln(x)\frac{x^6}{x} - \int \frac{x^6}{6}\frac{1}{x}\,dx = \frac{x^6\ln(x)}{6} - \frac{1}{6}\int x^5\,dx$

Integration by parts

$u = \ln(x) \qquad dv = x^5\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{x^6}{6}$

$$= \boxed{\frac{x^6\ln(x)}{6} - \frac{x^6}{36} + C}$$

5. Use integration by parts to find $\int \sin^{-1}(x)\,dx$ $= \sin^{-1}(x)\,x - \int x \frac{1}{\sqrt{1-x^2}}\,dx$

$u = \sin^{-1}(x)$ $\quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$ $\quad v = x$

$= x\sin^{-1}(x) + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$

$= x\sin^{-1}(x) + \frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + C$

$= \boxed{x\sin^{-1}(x) + \sqrt{1-x^2} + C}$

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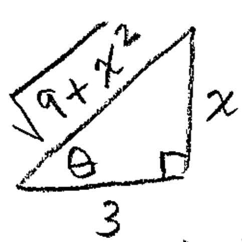
6. $\int \frac{1}{\sqrt{9+x^2}}\,dx = \int \frac{1}{\sqrt{9+(3\tan(\theta))^2}}\, 3\sec^2(\theta)\,d\theta$

$x = 3\tan(\theta)$

$dx = 3\sec^2(\theta)\,d\theta$

$\tan(\theta) = \frac{x}{3} = \frac{\text{OPP}}{\text{ADJ}}$



$= \int \frac{3\sec^2(\theta)}{\sqrt{9\sec^2(\theta)}}\,d\theta$

$= \int \frac{3\sec^2(\theta)}{3\sec(\theta)}\,d\theta$

$= \int \sec(\theta)\,d\theta$

$= \ln|\sec(\theta) + \tan(\theta)| + C$

$= \boxed{\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C}$

7. $\int \frac{2}{x^3 - x}\,dx = \int \frac{2}{x(x^2-1)}\,dx = \int \frac{2}{x(x-1)(x+1)}\,dx$

$$\frac{2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$x=0 \Rightarrow 2 = A(-1) \Rightarrow A = -2$

$x=1 \Rightarrow 2 = B\cdot 2 \Rightarrow B = 1$

$x=-1 \Rightarrow 2 = C\cdot 2 \Rightarrow C = 1$

$$= \int \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\,dx$$

$$= \int \frac{-2}{x} + \frac{1}{x-1} + \frac{1}{x+1}\,dx$$

$$= -2\ln|x| + \ln|x-1| + \ln|x+1| + C$$

$$= -\ln|x^2| + \ln|x-1| + \ln|x+1| + C$$

$$= \boxed{\ln\left|\frac{(x-1)(x+1)}{x^2}\right| + C}$$

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8. $\int_0^\infty \frac{e^x}{e^{2x}+1}\,dx = \lim_{b\to\infty} \int_0^b \frac{e^x}{(e^x)^2+1}\,dx$

$u = e^x$

$du = e^x\,dx$

$$= \lim_{b\to\infty} \int_{e^0}^{e^b} \frac{1}{u^2+1}\,du = \lim_{b\to\infty} \left[\tan^{-1}(u)\right]_1^{e^b}$$

$$= \lim_{b\to\infty} \left(\tan^{-1}(e^b) - \tan^{-1}(1)\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \boxed{\frac{\pi}{4}}$$

9. $\int \frac{1+\sin(x)+\cos(x)}{1+\sin(x)}\,dx = \int \frac{1+\sin(x)}{1+\sin(x)} + \frac{\cos(x)}{1+\sin(x)}\,dx$

$$= \int 1 + \frac{\cos(x)}{1+\sin(x)}\,dx = \int dx + \int \frac{\cos(x)}{1+\sin(x)}\,dx$$

$$= x + \int \frac{\cos(x)}{1+\sin(x)}\,dx$$

$u = 1+\sin(x)$

$du = \cos(x)dx$

$$= x + \int \frac{1}{u}\,du$$

$$= x + \ln|u| + C = \boxed{x + \ln|1+\sin(x)| + C}$$

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10. $\int x\sqrt{x-2}\,dx = \int (u+2)\sqrt{u}\,du = \int (u^1+2)u^{1/2}\,du$

$u = x-2$

$du = dx$

$x = u+2$

$$= \int u^{3/2} + 2u^{1/2}\,du$$

$$= \frac{u^{5/2}}{5/2} + 2\frac{u^{3/2}}{3/2} + C$$

$$= \frac{2\sqrt{u}^5}{5} + \frac{4\sqrt{u}^3}{3} + C$$

$$= \boxed{\frac{2\sqrt{x-2}^5}{5} + \frac{4\sqrt{x-2}^3}{3} + C}$$