MATH 211 — Test #1 ♣ — October 3, 2018

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Directions Except in a problem designated **short answer**, you must show your work to get full credit. This test is closed-book and closed-notes. No calculators or other electronic devices are allowed. Simplify your answers if it is easy to do so, but you may leave complex answers unsimplified, as in, for example, $7^{15} - 7!$. All you will need is something to write with. Scratch paper will be provided.

1. (9 points) **Short answer.**

(a) Write $\left\{\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \frac{25}{6}, \frac{36}{7}, \dots\right\}$ in set-builder notation. $= \boxed{\left\{\frac{n^2}{n+1} : n \in \mathbb{N}\right\}}$

(b) Write the set $\{2n^2 : n \in \mathbb{Z},\ \underbrace{0 \le 3n \le 9}\}$ by listing its elements between braces.

$\{n = 0, 1, 2, 3\}$ $\boxed{\{0, 2, 8, 18\}}$

(c) Write the set $\mathscr{P}(\{4\} \cap \mathbb{Z}) \times (\{0,1,2\} - \mathbb{N})$ by listing its elements between braces.

$= \mathscr{P}(\{4\}) \times \{0\} = \{\emptyset, \{4\}\} \times \{0\} = \boxed{\{(\emptyset, 0), (\{4\}, 0)\}}$

2. (12 points) **Short answer.** Suppose $A = \{1,3,4,6,9\}$ and $B = \{4,5,6,8,9\}$ are two sets in a universal set $U = \{1,2,3,4,5,6,7,8,9\}$

(a) $A \cap \overline{B} = \{1,3,4,6,9\} \cap \{1,2,3,7\} = \boxed{\{1,3\}}$

(b) $\overline{\emptyset} = U = \boxed{\{1,2,3,4,5,6,7,8,9\}}$

(c) $\overline{U} \times A = \emptyset \times A = \boxed{\emptyset}$

(d) $A - (A \times A) = A = \boxed{\{1,3,4,6,9\}}$

3. (9 points) Write a truth table for $P \Rightarrow Q$ and $\sim(P) \vee Q$. Based on this, say whether or not these two expressions are logically equivalent.

P	Q	$\sim P$	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Their columns are identical, so $P \Rightarrow Q$ and $(\sim P) \vee Q$ are logically equivalent.

4. (20 points) This question concerns **length-4** lists made from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9.

(a) How many such lists are there if **repetition is allowed** and there is least one repeated entry?

$$\underbrace{9\ 9\ 9\ 9}_{\text{all lists}} - \underbrace{9\cdot 8\cdot 7\cdot 6}_{\text{no repetition}} = 9^4 - 9\cdot 8\cdot 7\cdot 6 = \boxed{3537}$$

(Subtraction Principle)

(b) How many such lists are there if **repetition is allowed** and the last two entries are odd?

$$9\ 9\ \underbrace{5\ 5}_{\text{odd}} = 9^2 5^2 = \boxed{2025}$$

(c) How many such lists are there if **repetition is not allowed** and the last two entries are odd?

$$7\cdot 6\cdot \underbrace{5\cdot 4}_{\text{odd}} = P(7,4) = \boxed{840}$$

(d) How many such lists are there if **repetition is allowed** and the list has exactly two odd entries?

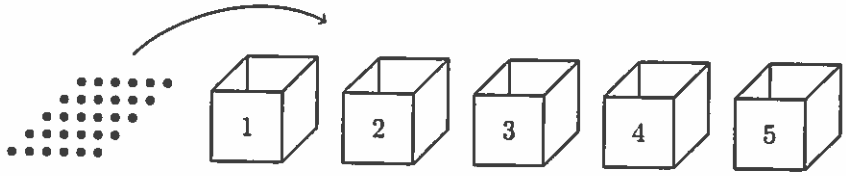
Such lists fall into six categories, or types:

o o e e	o e o e	o e e o	e o o e	e o e o	e e o o
5 5 4 4	5 4 5 4	5 4 4 5	4 5 5 4	4 5 4 5	4 4 5 5

By the addition principle, the answer is

$$6\cdot 5^2 4^2 = \boxed{2400}$$

5. (8 points) In how many ways can you put 30 identical balls in five different boxes?



$$\overbrace{**\cdots*}^{\text{Box 1}} \mid \overbrace{**\cdots*}^{\text{Box 2}} \mid \overbrace{***\cdots*}^{\text{Box 3}} \mid \overbrace{**\cdots*}^{\text{Box 4}} \mid \overbrace{**\cdots*}^{\text{Box 5}}$$

The number of ways to do this equals the number of lists of length $30+4=34$ consisting of 30 stars and 4 bars. Therefore the answer is

$$\binom{34}{4} = \frac{34!}{4!\,30!} = \frac{34\cdot33\cdot32\cdot31}{4\cdot3\cdot2} = \boxed{46{,}376}$$

6. (8 points) Consider the length-6 lists made from the symbols A, B, C, D, E, F, with **no repetition**. How many such lists have the property that the letters A, B, C occur in alphabetical order?
(For example, $ABCDEF$ or $ADBECF$, or $FEDABC$, but **not** $BACFED$, etc.)

	A		B		C

First, choose 3 out of 6 positions and put A, B, C in them, in order. There are $\binom{6}{3} = \frac{6!}{3!3!} = \frac{6\cdot5\cdot4}{6} = 20$ ways to do this.

	A		B		C
3		2		1	

Next, fill in the remaining positions in $3\cdot2\cdot1 = 6$ ways.

By multiplication principle, # of lists is $\boxed{\binom{6}{3}3! = 20\cdot6 = 120}$

7. (10 points)

(a) A row of Pascal's triangle begins as 1 8 28 56 70....
Write out the entire row. (You can do this without working out the triangle from top down.)

This row is

$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

$$\boxed{1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1}$$

(b) Using part (a) above, and the binomial theorem, expand $(x-1)^8$.
(Please simplify to the extent possible — it's easy to do!)

$$(x-1)^8 = \binom{8}{0}x^8 + \binom{8}{1}x^7(-1) + \binom{8}{2}x^6(-1)^2 + \binom{8}{3}x^5(-1)^3 + \binom{8}{4}x^4(-1)^4 + \binom{8}{5}x^3(-1)^5$$
$$+\binom{8}{6}x^2(-1)^6 + \binom{8}{7}x(-1)^7 + \binom{8}{8}(-1)^8$$

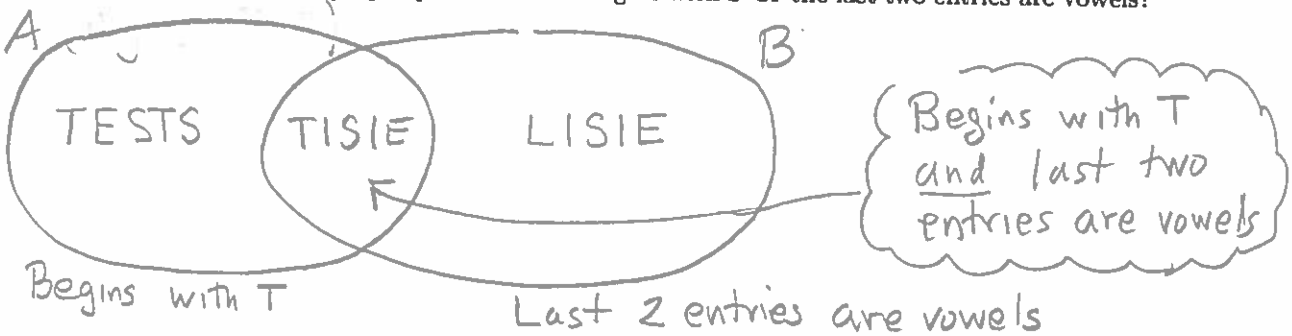
$$= \boxed{x^8 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1}$$

8. (8 points) Explain why $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4} + \binom{n}{5} + \cdots + \binom{n}{n}$.

By the Binomial Formula,

$$2^n = (1+1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}1 + \binom{n}{2}1^{n-2}1^2 + \binom{n}{3}1^{n-3}1^3 + \cdots + \binom{n}{n}1^n$$
$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

9. (8 points) Consider lists of length 5 made from the letters L, I, S, T, E, D, with **repetition allowed.** How many such lists have the property that the list begins with T or the last two entries are vowels?



$|A| = 1\cdot 6\cdot 6\cdot 6\cdot 6$

$|B| = 6\cdot 6\cdot 6\cdot 2\cdot 2$

$|A\cap B| = 1\cdot 6\cdot 6\cdot 2\cdot 2$

Answer $|A\cup B| = |A| + |B| - |A\cap B|$

$= 1\cdot 6^4 + 6^3\cdot 2^2 - 6^2 2^2 = \boxed{2016}$

10. (8 points) You deal seven cards from a 52-card deck and line them up in a row. How many possible 7-card lineups are there where no two cards of the same color are next to one another? (A deck has 26 red cards and 26 black cards.)

B	R	B	R	B	R	B

$26\cdot 26\cdot 25\cdot 25\cdot 24\cdot 24\cdot 23$

R	B	R	B	R	B	R

$26\cdot 26\cdot 25\cdot 25\cdot 24\cdot 24\cdot 23$

Two types of lineups.

By the addition principle, the answer is

$2\cdot(26^2 25^2 24^2\cdot 23) = \boxed{11{,}194{,}560{,}000}$