MATH 211 — Test #2 ♣ — November 16, 2017

Name: Richard — R. Hammack — Score: 100

Directions No calculators. Please put all phones, etc., away.

1. (4 points) Complete the following truth tables.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Q	R	$Q \Leftrightarrow R$
T	T	T
T	F	F
F	T	F
F	F	T

2. (12 points) Complete the truth table to decide if $P \Rightarrow (Q \wedge R)$ and $(\sim P) \vee (Q \Leftrightarrow R)$ are logically equivalent.

P	Q	R	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	$\sim P$	$(Q \Leftrightarrow R)$	$(\sim P) \vee (Q \Leftrightarrow R)$
T	T	T	T	T	F	T	T
T	T	F	F	F	F	F	F
T	F	T	F	F	F	F	F
T	F	F	F	F ←	F	T	T ←
F	T	T	T	T	T	T	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	T	T

Are they logically equivalent? Why or why not? The columns for $P \Rightarrow (Q \wedge R)$ and $(\sim P) \vee (Q \Leftrightarrow R)$ almost match, but not quite. Therefore they are NOT logically equivalent.

3. (6 points) Suppose the statement $(P \vee \sim P) \Leftrightarrow (P \wedge Q \wedge \sim R)$ is true. Find the truth values of P, Q and R. (This can be done without a truth table.)

Note that $(P \vee \sim P)$ is TRUE, so $(P \vee \sim P) \Leftrightarrow (P \wedge Q \wedge \sim R)$ being true means that $P \wedge Q \wedge \sim R$ is true. But $P \wedge Q \wedge \sim R$ being true means that P, Q and $\sim R$ are all true. Therefore

$$\boxed{P = T, \quad Q = T, \quad R = F}$$

4. (12 points) This problem concerns the following statement.
P: For each $n \in \mathbb{Z}$, there exists a number $m \in \mathbb{Z}$ for which $n + m = 0$.

(a) Is the statement P true or false? **Explain.**

This is true, because for any $n \in \mathbb{Z}$, let $m \in \mathbb{Z}$ be the number $m = -n$. Then $n+m = n-n = 0$

(b) Write the statement P in symbolic form.

$\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m+n = 0$

(c) Form the negation $\sim P$ of your answer from (b), and simplify.

$\sim(\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m+n=0)$
$= \exists n \in \mathbb{Z} \sim(\exists m \in \mathbb{Z}, m+n=0)$
$= \exists n \in \mathbb{Z}, \forall m \in \mathbb{Z} \sim(m+n=0)$
$= \boxed{\exists n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m+n \neq 0}$

(d) Write the negation $\sim P$ as an English sentence.
(The sentence may use mathematical symbols.)

There exists an integer n for which $m+n \neq 0$ for every integer m

5. (6 points) Complete the first and last lines of each of the following proof outlines.

Proposition: If P, then Q. **Proof:** (Direct) Suppose P ⋮ Therefore Q. ■	**Proposition:** If P, then Q. **Proof:** (Contrapositive) Suppose $\sim Q$ ⋮ Therefore $\sim P$. ■	**Proposition:** If P, then Q. **Proof:** (Contradiction) Suppose $P \wedge \sim Q$ ⋮ Therefore $C \wedge \sim C$. ■

6. (15 points) Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$.
Prove: If $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{n}$. [Use direct proof.]

Proof (Direct) Suppose $a \equiv b \pmod{n}$
This means $n \mid (a-b)$ and consequently
$a - b = nk$ for some $k \in \mathbb{Z}$,

Now multiply both sides by $a+b$:

$$a-b = nk$$
$$(a+b)(a-b) \quad nk(a+b)$$
$$a^2 - b^2 = nk(a+b).$$

Therefore $a^2 - b^2 = nc$ for $c = k(a+b) \in \mathbb{Z}$.
Consequently $n \mid a^2 - b^2$.
Therefore $a^2 \equiv b^2 \pmod{n}$.

7. (15 points) Suppose $a \in \mathbb{Z}$. **Prove:** If $100 \nmid a^2$, then a is odd or $5 \nmid a$. [Use contrapositive.]

Proof (Contrapositive).
Suppose it is not true that a is odd or $5 \nmid a$.
Then a is even and $5 \mid a$.
Therefore $\boxed{a = 2c}$ for some $c \in \mathbb{Z}$,
and $\boxed{a = 5d}$ for some $d \in \mathbb{Z}$.
This means $2c = 5d$, so $5d$ is even. But then d must be even because if it were odd, then $5d$ would be odd not even. Because d is even we get $d = 2e$ for some $e \in \mathbb{Z}$,
Thus $a = 5d = 5 \cdot 2e = 10e$.
Consequently $a^2 = (10e)^2 = 100e^2$.
As $a^2 = 100k$ for $k = e^2$ we obtain $100 \mid a^2$.
Hence it is not true that $100 \nmid a^2$

8. (15 points) **Prove:** If $a \in \mathbb{Z}$, then $4 \nmid (a^2 - 3)$. [Contradiction may be easiest.]

Proof Suppose for the sake of contradiction that $4 \mid (a^2 - 3)$. This means $\boxed{a^2 - 3 = 4k}$ for some $k \in \mathbb{Z}$. From this,

$$a^2 = 4k + 3 = 4k + 2 + 1 = 2(2k+1) + 1$$

and therefore a^2 is odd. Hence a is also odd, so $a = 2l + 1$ for some $l \in \mathbb{Z}$.

Now we have

$$\begin{aligned} a^2 - 3 &= 4k \\ (2l+1)^2 - 3 &= 4k \\ 4l^2 + 4l + 1 - 3 &= 4k \\ 4l^2 + 4l - 2 &= 4k \\ 4l^2 + 4l - 4k &= 2 \\ 2(l^2 + l - k) &= 1 \end{aligned}$$

(divide by 2, then factor out a 2 on left)

Consequently 1 is even, which is a contradiction ∎

9. (15 points) **Prove:** If $n \in \mathbb{N}$, then $1 + (-1)^n(2n - 1)$ is a multiple of 4. [Try cases.]

Proof (Direct) Suppose $n \in \mathbb{N}$.

CASE I If n is even, then $n = 2c$ and $(-1)^n = 1$. Then $1 + (-1)^n(2n-1) = 1 + 1 \cdot (2(2c) - 1) = 1 + 4c - 1 = 4c$, and this is a multiple of 4.

CASE II If n is odd, then $n = 2c + 1$ and $(-1)^n = -1$. Then $1 + (-1)^n(2n-1) = 1 + (-1)(2(2c+1) - 1) = 1 - (4c + 2 - 1) = 1 - 4c - 2 + 1 = -4c$, and this is a multiple of 4.

So in either case, $1 + (-1)^n(2n-1)$ is a multiple of 4 ∎