MATH 211 Test #1 ♠ October 3, 2018

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Directions Except in a problem designated **short answer**, you must show your work to get full credit. This test is closed-book and closed-notes. No calculators or other electronic devices are allowed. Simplify your answers if it is easy to do so, but you may leave complex answers unsimplified, as in, for example, $7^{15} - 7!$. All you will need is something to write with. Scratch paper will be provided.

1. (9 points) **Short answer.**

   (a) Write $\{\ldots -13, -3, 7, 17, 27, 37, \ldots\}$ in set-builder notation.

   $\boxed{\{7+10n : n \in \mathbb{Z}\}}$

   (b) Write the set $\{X : X \subseteq \{1,2,3,4\}, |X| \geq 3\}$ by listing its elements between braces.

   $\boxed{\{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}}$

   (c) Write the set $\mathscr{P}(\{-1,0,1,2\} - \mathbb{N})$ by listing its elements between braces.

   $= \mathscr{P}(\{-1,0\}) = \boxed{\{\emptyset, \{-1\}, \{0\}, \{-1,0\}\}}$

2. (12 points) **Short answer.** Suppose $X = \{a,b,c,d,e\}$ and $Y = \{d,e,f,g\}$ are two sets in a universal set $U = \{a,b,c,d,e,f,g,h\}$.

   (a) $X \cap \overline{Y} = \{a,b,c,d,e\} \cap \{a,b,c,h\} = \boxed{\{a,b,c\}}$

   (b) $\overline{X \cup \overline{Y}} = \overline{\{a,b,c,d,e,h\}} = \boxed{\{f,g\}}$

   (c) $X - \mathscr{P}(X) = \boxed{X = \{a,b,c,d,e\}}$ (no element of X is an element of its power set)

   (d) $(X \cap Y) \times (Y - X) = \{d,e\} \times \{f,g\} = \boxed{\{(d,f),(d,g),(e,f),(e,g)\}}$

3. (9 points) Write a truth table for $P \Leftrightarrow \sim Q$ and $\sim(P \Leftrightarrow Q)$. Based on this, say whether or not these two expressions are logically equivalent.

P	Q	$\sim Q$	$P \Leftrightarrow \sim Q$	$P \Leftrightarrow Q$	$\sim(P \Leftrightarrow Q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	T	F	T
F	F	T	F	T	F

The columns for $P \Leftrightarrow \sim Q$ and $\sim(P \Leftrightarrow Q)$ are identical, so these expressions are logically equivalent.

4. (20 points) This question concerns **length-4** lists made from the letters A, B, C, D, E, F.

(a) How many such lists are there in which repetition is not allowed?

$$\underline{6}\ \underline{5}\ \underline{4}\ \underline{3}$$

Ans. $P(6,4) = 6 \cdot 5 \cdot 4 \cdot 3 = \boxed{360}$

(b) How many such lists are there if **repetition is allowed**, and exactly one entry is a vowel?

$$\underline{V}\ \underline{\ }\ \underline{\ }\ \underline{\ } \qquad \underline{\ }\ \underline{V}\ \underline{\ }\ \underline{\ } \qquad \underline{\ }\ \underline{\ }\ \underline{V}\ \underline{\ } \qquad \underline{\ }\ \underline{\ }\ \underline{\ }\ \underline{V}$$

$$2\,4\,4\,4 + 4\,2\,4\,4 + 4\,4\,2\,4 + 4\,4\,4\,2$$

$$= 4 \cdot 2 \cdot 4^3 = 4^4 \cdot 2 = 2^9 = \boxed{512}$$

(c) How many such lists are there if **repetition is allowed**, and there is at least one repeated entry?

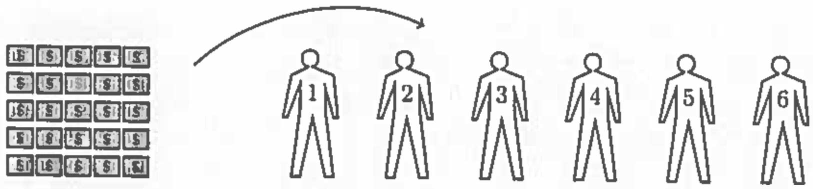
$$\underbrace{6 \cdot 6 \cdot 6 \cdot 6}_{\text{all lists}} - \underbrace{6\ 5\ 4\ 3}_{\text{no repetition}} = 6^4 - 6 \cdot 5 \cdot 4 \cdot 3 = \boxed{936}$$

(d) How many such lists are there if repetition is **not allowed**, and the list is in alphabetical order?

$$\binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{4 \cdot 3 \cdot 2 \cdot 2} = \boxed{15}$$

choose 4 out of 6 letters, then arrange them in alphabetical order.

5. (8 points) You are going to distribute 25 one-dollar bills among six friends. (So each friend could potentially get between 0 and 25 dollars.) In how many ways can this be done?



person 1 | person 2 | person 3 | person 4 | person 5 | person 6

$\underbrace{* \cdots *}$ | $\underbrace{* \cdots *}$ | $\underbrace{* \cdots *}$ | $\underbrace{* \cdots *}$ | $\underbrace{* \cdots *}$ | $\underbrace{* \cdots *}$

There is a star for each dollar a particular person gets, and 5 bars dividing the groups of stars. Such a list has length $25+5=30$. To make such a list, choose 5 out of 30 positions for bars and fill the rest with stars.

Answer $\binom{30}{5} = \frac{30!}{5!\,25!} = \frac{30\cdot29\cdot28\cdot27\cdot26}{5\cdot4\cdot3\cdot2} = \boxed{142506}$

6. (8 points) This problem concerns length-6 lists made from the symbols A, B, C, D, E, F, G, H, with **no repetition**. How many such lists have two consecutive vowels?

(For example, $AECDBF$ or $CHEADF$, but **not** $BCADFE$, etc.)

V	V				
2	1	6	5	4	3

	V	V			
6	2	1	5	4	3

		V	V		
6	5	2	1	4	3

			V	V	
6	5	4	2	1	3

				V	V
6	5	4	3	2	1

Answer: By addition principle:

$5\cdot(2\cdot1)(6\cdot5\cdot4\cdot3) = 5\cdot6! = \boxed{3600}$

7. ($\cancel{8}$ 10 points)

(a) A row of Pascal's triangle begins as 1 8 28 56 70....
Write out the entire row. (You can do this without working out the triangle from top down.)

This row is:

$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

(b) Using part (a) above, and the binomial theorem, expand $(x-1)^8$.
(Please simplify to the extent possible — it's easy to do!)

$$(x-1)^8 =$$

$$\binom{8}{0}x^8 + \binom{8}{1}x^7(-1) + \binom{8}{2}x^6(-1)^2 + \binom{8}{3}x^5(-1)^3 + \binom{8}{4}x^4(-1)^4 + \binom{8}{5}x^3(-1)^5 + \binom{8}{6}x^2(-1)^6 + \binom{8}{7}x(-1)^7 + \binom{8}{8}(-1)^8$$

$$= \boxed{x^8 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1}$$

8. (8 points) Explain why $4^n = \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + 3^3\binom{n}{3} + 3^4\binom{n}{4} + 3^5\binom{n}{5} + \cdots + 3^n\binom{n}{n}$.

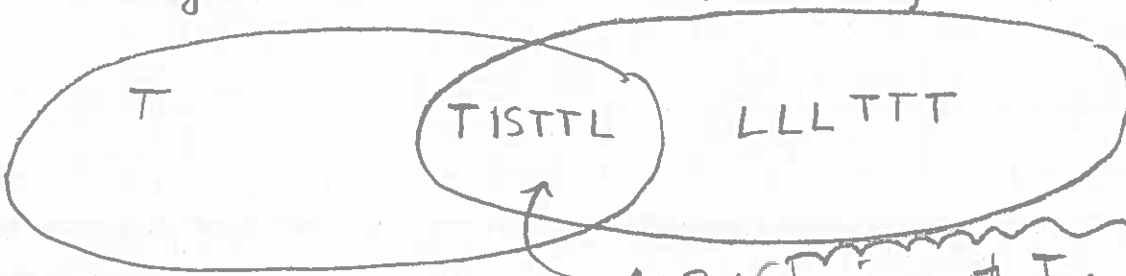
By binomial formula:

$$4^n = (1+3)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}3^1 + \binom{n}{2}1^{n-2}3^2 + \binom{n}{3}1^{n-3}3^3 + \cdots + \binom{n}{n}3^n$$

$$= \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + 3^3\binom{n}{3} + \cdots + 3^n\binom{n}{n}$$

9. (~~10~~ 8 points) Consider lists of length 6 made from the letters L, I, S, T, E, D, with **repetition allowed.** How many such lists have the property that the list begins with T or the list has exactly three T's?

A: Begins with T B: Exactly 3 T's



$A \cap B$: (begins with T and exactly 3 T's.)

A: | T | | | | | |
6 6 6 6 6

$|A| = 6^5$

B: | | T | | T | T | |
5 5 5

$|B| = \binom{6}{3} 5^3$

$A \cap B$: | T | | T | | T | |
5 5 5

$|A \cap B| = \binom{6}{2} 5^3$

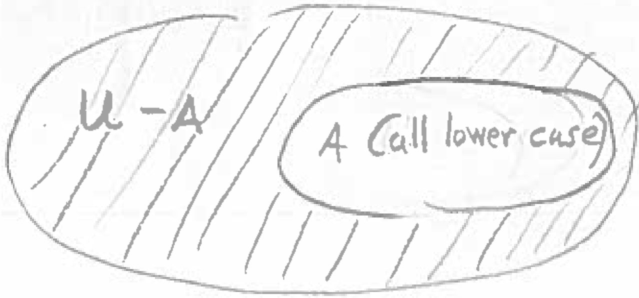
Ans:

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 6^5 + \binom{6}{3} 5^3 - \binom{6}{2} 5^3$

$= 6^5 + 5^3\left(\binom{6}{3} - \binom{6}{2}\right) = \boxed{8401}$

10. (8 points) A password for a certain account must be 7 characters long, consisting of letters of the English alphabet (upper or lower case, but no numbers or special symbols). Also the password cannot be all lower case. How many such passwords are possible?



U = all passwords

A = (all lower-case passwords)

U: | | | | | | | |
52 52 52 52 52 52 52

$|U| = 56^7$

A | | | | | | | |
26 26 26 26 26 26 26

$|A| = 26^7$

Answer: $\boxed{52^7 - 26^7}$