Section 14.4

(4) $w = \ln(x^2 + y^2 + z^2)$ $\begin{cases} x = \cos t \\ y = \sin t \\ z = 4\sqrt{t} \end{cases}$

(a) Method I

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{2x}{x^2+y^2+z^2}(-\sin t) + \frac{2y}{x^2+y^2+z^2}(\cos t) + \frac{2z}{x^2+y^2+z^2}\, 2\frac{1}{\sqrt{t}}$$

$$= \frac{-2x\sin t + 2y\cos t + 4z\frac{1}{\sqrt{t}}}{x^2 + y^2 + z^2}$$

$$= \frac{-2\cos t\sin t + 2\sin t\cos t + 2\cdot 4\sqrt{t}\, 2\frac{1}{\sqrt{t}}}{\cos^2 t + \sin^2 t + (4\sqrt{t})^2}$$

$$= \boxed{\frac{16}{1+16t}}$$

Method II

$$w = \ln\left(\cos^2 t + \sin^2 t + (4\sqrt{t})^2\right)$$

$$= \ln(1 + 16t)$$

$$\frac{dw}{dt} = \boxed{\frac{16}{1+16t}}$$

(b) $$\left.\frac{dw}{dt}\right|_{t=3} = \frac{16}{1+16\cdot 3} = \boxed{\frac{16}{49}}$$

Section 14.4

(8) $z = \tan^{-1}\left(\frac{x}{y}\right)$ $\quad x = u\cos v \quad y = u \sin v$

(a)
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}\cos v + \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)\sin v$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{\cos v}{y} - \frac{x \sin v}{y^2}\right)$$

$$= \frac{1}{1+\left(\frac{u\cos v}{u \sin v}\right)^2}\left(\frac{\cos v}{u \sin v} - \frac{u\cos v \sin v}{u^2 \sin^2 v}\right)$$

$$= \frac{1}{1+\cot^2 v}\left(\frac{1}{u}\cot v - \frac{1}{u}\cot v\right) = \boxed{0}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}(-u\sin v) + \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)u\cos v$$

$$= \frac{1}{1+\left(\frac{u\cos v}{u\sin v}\right)^2}\left(\frac{-u\sin v}{u\sin v} - \frac{u\cos v\, u\cos v}{u^2\sin^2 v}\right)$$

$$= \frac{1}{1+\cot^2 v}\left(-1 - \cot^2 v\right)$$

$$= -\frac{1+\cot^2 v}{1+\cot^2 v} = \boxed{-1}$$

Recall identity $1+\cot^2\theta = \csc^2\theta$.

Method II $\quad z = \tan^{-1}\left(\frac{u\cos v}{u \sin v}\right) = \tan^{-1}(\cot v)$

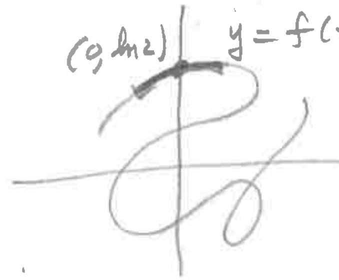
$$\frac{\partial z}{\partial u} = \boxed{0} \qquad \frac{\partial z}{\partial v} = \frac{1}{1+\cot^2 v}\left(-\csc^2 v\right) = \frac{-1-\cot^2 v}{1-\cot^2 v} = \boxed{-1}$$

(8) (Continued)

$$\left.\frac{\partial z}{\partial u}\right|_{(1.3,\pi/6)} = \boxed{0} \qquad \left.\frac{\partial z}{\partial v}\right|_{(1.3,\pi/6)} = \boxed{-1}$$

(28) $xe^y + \sin xy + y - \ln 2 = 0$ at $(0, \ln 2)$

This equation defines an implicit function $y=f(x)$ whose graph crosses the point $(0, \ln 2)$. See diagram.



Need to find $\frac{dy}{dx}$ and plug in $(0, \ln 2)$

Let $F(x,y) = xe^y + \sin xy + y - \ln 2$

Thus the above equation is now $F(x,y)=0$

Then $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$

$\Rightarrow \left(e^y + \cos(xy)y\right)\cdot 1 + \left(xe^y + \cos(xy)x + 1\right)\frac{dy}{dx} = 0$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-e^y - \cos(xy)y}{xe^y + \cos(xy)x + 1}}$$

$$\left.\frac{dy}{dx}\right|_{(0,\ln 2)} = \frac{-e^{\ln 2} - \cos(0\cdot\ln 2)\ln 2}{0e^{\ln 2} + \cos(0\cdot\ln 2)\cdot 0 + 1} = \frac{-2-\ln 2}{1}$$

$$= \boxed{-2-\ln 2}$$

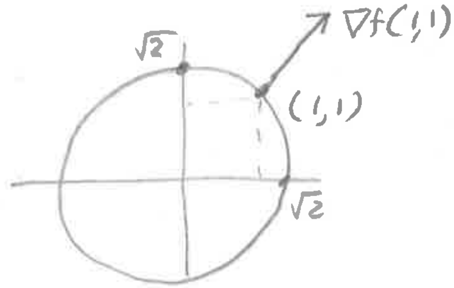
Section 14.5

(2) $f(x,y) = \ln(x^2+y^2)$ at $(1,1)$

$$\nabla f(x,y) = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$$

$$\nabla f(1,1) = \left\langle \frac{2}{1+1}, \frac{2}{1+1} \right\rangle = \boxed{\langle 1, 1 \rangle}$$

At $(1,1)$ we have $f(1,1) = \ln(1^2+1^2) = \ln 2$. Consider the level curve passing through this point, i.e. $f(x,y) = \ln 2$, or $\ln(x^2+y^2) = \ln 2$. Then $x^2+y^2 = 2$, so this curve is the circle of radius $\sqrt{2}$ centered at the origin, and indeed $(1,1)$ is on this curve.



(10) $f(x,y,z) = e^{x+y}\cos(z) + (y+1)\sin^{-1}(x)$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \left\langle e^{x+y}\cos z + \frac{y+1}{\sqrt{1-x^2}},\ e^{x+y}\cos z + \sin^{-1}x,\ -e^{x+y}\sin z \right\rangle$$

$$\nabla f(0,0,\pi/6) = \left\langle e^0\cos\frac{\pi}{6} + \frac{0+1}{\sqrt{1-0^2}},\ e^0\cos\frac{\pi}{6} + \sin^{-1}0,\ -e^0\sin\frac{\pi}{6} \right\rangle$$

$$= \left\langle \frac{\sqrt{3}}{2}+1,\ \frac{\sqrt{3}}{2},\ -\frac{1}{2} \right\rangle$$

Section 14.5

(16) Find directional derivative of $f(x,y,z) = x^2+2y^2-3z^2$ in the direction of $\langle 1,1,1\rangle$ at $P_0(1,1,1)$

The unit vector in the given direction is $\vec{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$.

Answer: $D_{\vec{u}}(f) = \nabla f \cdot \vec{u}$

$$= \langle 2x, 4y, -6z\rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle$$

$$= \frac{2x}{\sqrt{3}} + \frac{4y}{\sqrt{3}} - \frac{6z}{\sqrt{3}}$$

Then $D_{\vec{u}}(f)(1,1,1) = \frac{2}{\sqrt{3}} + \frac{4}{\sqrt{3}} - \frac{6}{\sqrt{3}} = \boxed{0}$

(22) $g(x,y,z) = xe^y + z^2$ at $(1, \ln 2, \frac{1}{2})$

Direction of greatest increase at (x,y,z) is

$\nabla g = \langle e^y, xe^y, 2z\rangle$

Direction of greatest increase at $(1, \ln 2, \frac{1}{2})$ is

$\nabla g(1, \ln 2, \frac{1}{2}) = \boxed{\langle 2, 2, 1\rangle} \xrightarrow{\text{normalize}} \boxed{\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle}$

Direction of greatest decrease at $(1, \ln 2, \frac{1}{2})$ is

$-\nabla g(1, \ln 2, \frac{1}{2}) = \boxed{\langle -2, -2, -1\rangle} \xrightarrow{\text{normalize}} \boxed{\langle -\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\rangle}$

Rate of change of $g(x,y,z)$ in the direction of $\vec{u} = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$ is $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$$= \langle e^y, xe^y, 2z\rangle \cdot \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle = \frac{2e^y}{3} + \frac{2xe^y}{3} + \frac{2z}{3}.$$

At $(1, \ln 2, \frac{1}{2})$ this is $\frac{2e^{\ln 2}}{3} + \frac{2\cdot 1 e^{\ln 2}}{3} + \frac{2\cdot \frac{1}{2}}{3} = \frac{9}{3} = \boxed{3}$