MATH 307 §14.6

(12) Find the tangent plane to $f(x,y) = 4x^2 + y^2$ at $(1,1,5)$.

$f_x(x,y) = 8x \qquad f_x(1,1) = 8$

$f_y(x,y) = 2y \qquad f_y(1,1) = 2$

Equation of tangent plane is

$$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$z = 5 + 8(x-1) + 2(y-1)$$

$$z = 5 + 8x - 8 + 2y - 2$$

$$\boxed{z = 8x + 2y - 5}$$

Equation for tangent plane at $(1,1)$.

Section 14.7

(14) Find local max/min/saddle points of $f(x,y) = x^3 + 3xy + y^3$

$$\nabla f = \langle 3x^2 + 3y,\ 3y^2 + 3x \rangle = 3\langle x^2 + y,\ y^2 + x \rangle = \langle 0, 0 \rangle$$

To find the critical points we need to solve the system

$$\begin{cases} x^2 + y = 0 \\ y^2 + x = 0 \end{cases} \Rightarrow x^2 + y - (y^2 + x) = 0$$

$$\Rightarrow x^2 - y^2 - (x - y) = 0$$

$$\Rightarrow (x-y)(x+y) - (x-y) = 0$$

$$\Rightarrow \underbrace{(x-y)}_{x=y}\underbrace{(x+y-1)}_{} = 0$$

Because $x^2 + y = 0$, y is not positive. Because $y^2 + x = 0$, x is not positive. Hence this term is negative, i.e. not 0

Putting $x = y$ into $x^2 + y = 0$ gives $y^2 + y = 0 \Rightarrow y(y+1) = 0$

$y = 0$, $x = 0$ $\quad$ $y = -1$, $y = -1$

Critical Points $\boxed{(0, 0) \text{ and } (-1, -1)}$

Need the following:

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = 3$

Point $(0,0)$: $\quad f_{xx} f_{yy} - f_{xy}^2 = 0 \cdot 0 - 3^2 = -9 < 0$

Thus there is a saddle point at $(0,0)$

Point $(-1,-1)$: $\quad f_{xx} f_{yy} - f_{xy}^2 = (-6)(-6) - 3^2 = 36 - 9 = 27 > 0$

Then because $f_{xx} = -6 < 0$ there is a local maximum at $(-1,-1)$

(24) $f(x,y) = e^{2x} \cos y \quad \nabla f = \langle 2e^{2x} \cos y,\ -e^{2x} \sin y \rangle = \langle 0, 0 \rangle$

Since $e^{2x} > 0$, we can only have $\nabla f = \langle 0, 0 \rangle$ for those y for which $\cos y = 0$ and $\sin y = 0$. As no such y exist there are no critical points, hence $\boxed{\text{no extrema.}}$

(28) Find local max/min and saddle points of

$f(x,y) = e^{x}(x^2-y^2)$

$\nabla f = \langle e^{x}(x^2-y^2) + e^{x}2x, -e^{x}2y \rangle$

$= \langle e^{x}(x^2+2x-y^2), -2ye^{x} \rangle = \langle 0, 0 \rangle$

$$\Rightarrow \begin{cases} e^{x}(x^2+2x-y^2) = 0 \\ -2ye^{x} = 0 \end{cases}$$

Because $e^{x} > 0$, $-2ye^{x} = 0$ implies $\boxed{y = 0}$

Then we have $e^{x}(x^2+2x-0^2) = 0$

$e^{x}(x^2+2x) = 0$

$e^{x}(x+2)x = 0$

$\boxed{x = -2 \quad x = 0}$

Critical points are $\boxed{(-2,0) \text{ and } (0,0)}$

$f_{xx}(x,y) = e^{x}(x^2+2x-y) + e^{x}(2x+2) = e^{x}(x^2+4x+2-y)$

$f_{yy}(x,y) = -2e^{x}$

$f_{yx}(x,y) = -2ye^{x}$

Point $(0,0)$ $f_{xx}f_{yy} - f_{xy}^2 = 2(-2)+0 = -4 < 0$

$\boxed{\text{Saddle point at } (0,0)}$

Point $(-2,0)$ $f_{xx}f_{yy} - f_{xy}^2 = e^{-2}(4-8+2-0)(-2e^{-2}) - 0$

$= 2e^{-4} > 0$

Also $f_{xx} = -2e^{-2} < 0$

$\boxed{\text{Local Maximum at } (-2,0)}$