Math 307 Section 14.8

⑧ Find the points on the curve $x^2 + xy + y^2 = 1$ that are nearest and farthest from the origin.

The distance of (x,y) from the origin is $\sqrt{x^2+y^2}$ but to find (x,y) that minimizes or maximizes this, we just need to find those (x,y) which minimize or maximize its square $f(x,y) = x^2 + y^2$.

Thus we seek the max / min of

$f(x,y) = x^2 + y^2$

subject to the constraint

$$\underbrace{x^2+xy+y^2-1}_{g(x,y)} = 0$$

Need to solve $\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 0 \end{cases}$

$$\Rightarrow \begin{cases} \langle 2x, 2y \rangle = \lambda \langle 2x+y, x+2y \rangle \\ x^2+xy+y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} 2x = \lambda(2x+y) & ① \\ 2y = \lambda(x+2y) & ② \\ x^2+xy+y^2 = 1 & ③ \end{cases}$$

Note: If $\lambda = 0$, ① and ② would give $x=0$ and $y=0$ and then putting this into $x^2+xy+y^2=1$, we'd get $0=1$. Therefore we conclude $\lambda \neq 0$

$$\Rightarrow \begin{cases} 2x = 2\lambda x + \lambda y \\ 2y = \lambda x + 2\lambda y \\ x^2+xy+y^2 = 1 \end{cases} \Rightarrow \begin{cases} 2xy = 2\lambda xy + \lambda y^2 & ⓐ \\ 2xy = \lambda x^2 + 2\lambda xy & ⓑ \\ x^2+xy+y^2 = 1 & ⓒ \end{cases}$$

$$ⓐ - ⓑ \Rightarrow 0 = \lambda x^2 - \lambda y^2 \Rightarrow \lambda x^2 = \lambda y^2 \Rightarrow x^2 = y^2$$

$$\Rightarrow \boxed{x = \pm y.}$$

If $\boxed{x=y}$ then ⓒ gives $x^2 + x^2 + x^2 = 1 \Rightarrow 3x^2 = 1 \Rightarrow x^2 = \frac{1}{3}$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$ and we get points $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

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On the other hand, if $\boxed{x=-y}$, then (c) gives $x^2+x(-x)+x^2=1$, or $x^2=1$, so $x=\pm 1$

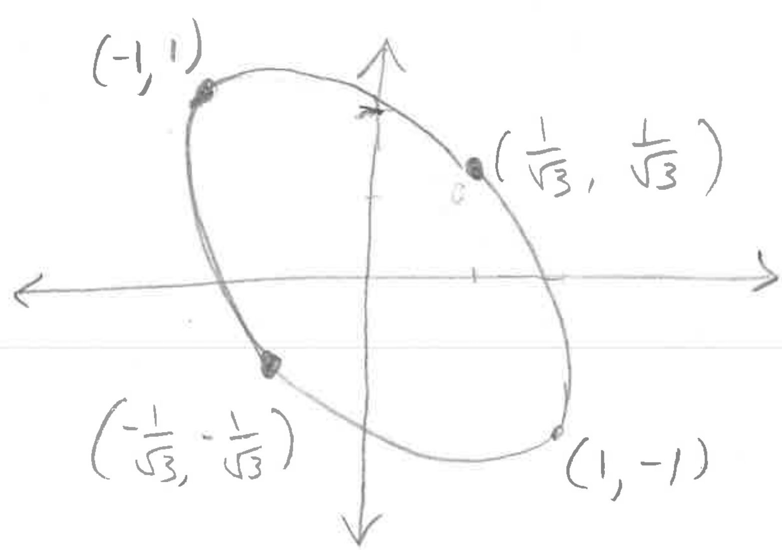
Then we get points $(1,-1)$ and $(-1,1)$.

$$\left.\begin{array}{l} f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{3}+\frac{1}{3} = \frac{2}{3} \\ f\left(-\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) = \frac{1}{3}+\frac{1}{3} = \frac{2}{3} \end{array}\right\} \text{minimum}$$

$$\left.\begin{array}{l} f(1,-1) = 1+1 = 2 \\ f(-1,1) = 1+1 = 2 \end{array}\right\} \text{maximum}$$

Answer:

The points $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ on $x^2+xy+y^2=1$ are at a minimum distance from the origin and $(1,-1)$ and $(-1,1)$ are at a maximum distance



(24) Find the points on the sphere $x^2+y^2+z^2=25$ where $f(x,y,z)=x+2y+3z$ has its max and min values.

Want to find max / min of

$f(x,y,z) = x + 2y + 3z$

Subject to $\underbrace{x^2+y^2+z^2-25}_{g(x,y,z)} = 0$

Set up system $\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 0 \end{cases}$

$$\Longrightarrow \begin{cases} \langle 1, 2, 3\rangle = \lambda\langle 2x, 2y, 2z\rangle \\ x^2+y^2+z^2-25=0 \end{cases}$$

$$\Longrightarrow \begin{cases} (1)\ 1 = 2\lambda x \\ (2)\ 2 = 2\lambda y \\ (3)\ 3 = 2\lambda z \\ (4)\ x^2+y^2+z^2 = 25 \end{cases}$$

Note: (1), (2), (3) say none of λ, x, y nor z is equal to zero

$$\frac{(2)}{(1)} \Longrightarrow 2 = \frac{2\lambda y}{2\lambda x} \Longrightarrow 2 = \frac{y}{x} \Longrightarrow \boxed{y = 2x}$$

$$\frac{(3)}{(1)} \Longrightarrow 3 = \frac{2\lambda z}{2\lambda x} \Longrightarrow 3 = \frac{z}{x} \Longrightarrow \boxed{z = 3x}$$

From $x^2 + y^2 + z^2 = 25$ we get

$$x^2 + (2x)^2 + (3x^2) = 25$$
$$x^2 + 4x^2 + 9x^2 = 25$$
$$14x^2 = 25$$
$$x^2 = \frac{25}{14}$$
$$\boxed{x = \pm \frac{5}{\sqrt{14}}}$$

Get points $\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right)$

and $\left(\frac{-5}{\sqrt{14}}, \frac{-10}{\sqrt{14}}, -\frac{15}{\sqrt{14}}\right)$

$$f\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right) = \frac{5}{\sqrt{14}} + 2\frac{10}{\sqrt{14}} + 3\frac{15}{\sqrt{14}} = \frac{70}{\sqrt{14}}$$
$$f\left(-\frac{5}{\sqrt{14}}, -\frac{10}{\sqrt{14}}, \frac{-15}{\sqrt{14}}\right) = -\frac{5}{\sqrt{14}} - 2\frac{10}{\sqrt{14}} - 3\frac{15}{\sqrt{14}} = -\frac{70}{\sqrt{14}}$$

Conclusion:

Minimum at $\left(-\frac{5}{\sqrt{14}}, \frac{-10}{\sqrt{14}}, \frac{-15}{\sqrt{14}}\right)$

Maximum at $\left(\frac{5}{\sqrt{14}}, \frac{10}{\sqrt{14}}, \frac{15}{\sqrt{14}}\right)$