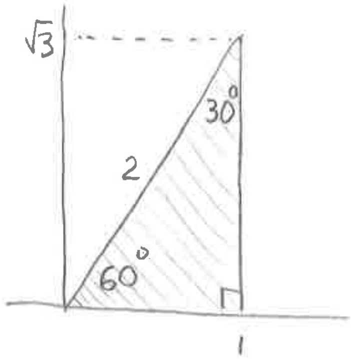
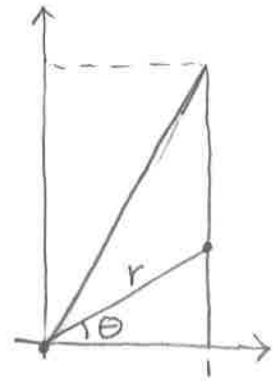
Section 15.4

(4)



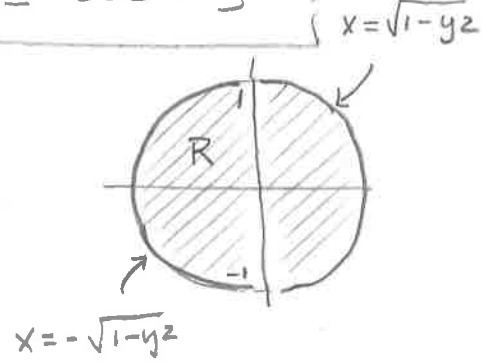
We recognize this as a 30-60-90 triangle, so for this region, $0 \le \theta \le \frac{\pi}{3}$



Also, for any θ we have $\cos\theta = \frac{\text{Adj}}{\text{hyp}} = \frac{1}{r}$ so that $r = \frac{1}{\cos\theta} = \sec\theta$

Answer: This is the polar region
$R = \{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{3},\ 0 \le r \le \sec\theta\}$

(20) $$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy$$



From the integral we can determine that R is bounded by the unit circle, i.e. graph of $x=-\sqrt{1-y^2}$ on the left and $x=\sqrt{1-y^2}$ on the right. In terms of polar coordinates this is the set of points $R=\{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1\}$
Therefore the above integral equals

$$\int_0^{2\pi}\int_0^1 \ln\left((r\cos\theta)^2 + (r\sin\theta)^2 + 1\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \ln(r^2+1)\,r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \ln(r^2+1)\,2r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\Big[(r^2+1)\ln(r^2+1) - (r^2+1)\Big]_0^1 d\theta$$

Make substitution $u = r^2+1$ and recall $\int \ln u\,du = u\ln u - u$ via integration by parts

$$= \frac{1}{2}\int_0^{2\pi} (2\ln 2 - 2) - (1\cdot\ln 1 - 1)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (2\ln 2 - 1)\,d\theta =$$

$$= \frac{2\ln 2 - 1}{2}\int_0^{2\pi} d\theta = \frac{2\ln 2 - 1}{2}\Big[\theta\Big]_0^{2\pi} = \boxed{\pi(2\ln 2 - 1)}$$

Section 15.4

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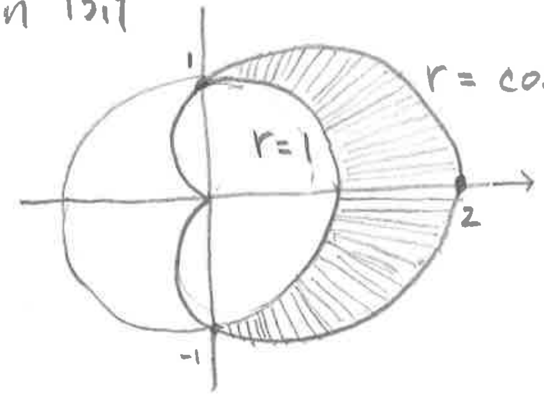

This region lies between polar angles $\theta = \frac{-\pi}{2}$ and $\theta = \frac{\pi}{2}$, and between graphs of $r=1$ & $r = \cos\theta + 1$

$$A = \iint_R 1\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{\cos\theta + 1} 1\, r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_1^{\cos\theta + 1} d\theta$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (\cos\theta + 1)^2 - 1\, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2\theta + 2\cos\theta\, d\theta$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2\theta}{2} + 2\cos\theta\, d\theta$$

$$= \frac{1}{2} \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{0}{4} + 2\right) - \left(-\frac{\pi}{4} + \frac{0}{4} - 2\right)\right] = \frac{1}{2}\left(\frac{\pi}{2} + 4\right)$$

$$= \boxed{\frac{\pi}{4} + 2 \text{ square units}}$$

Section 15.5

(16) $$\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} \Big[xz \Big]_3^{4-x^2-y} dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} x(1-x^2-y) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} x - x^3 - xy \, dy \, dx$$

$$= \int_0^1 \Big[xy - x^3 y - \frac{x y^2}{2} \Big]_0^{1-x^2} dx$$

$$= \int_0^1 x(1-x^2) - x^3(1-x^2) - \frac{1}{2} x (1-x^2)^2 \, dx$$

$$= \int_0^1 x - x^3 - x^3 + x^5 - \frac{x}{2} + x^3 - \frac{x^5}{2} \, dx$$

$$= \int_0^1 \frac{1}{2} x - x^3 + \frac{x^5}{2} \, dx$$

$$= \Big[\frac{x^2}{4} - \frac{x^4}{4} + \frac{x^6}{12} \Big]_0^1 = \frac{1}{4} - \frac{1}{4} + \frac{1}{12} = \boxed{\frac{1}{12}}$$

Section 15.5

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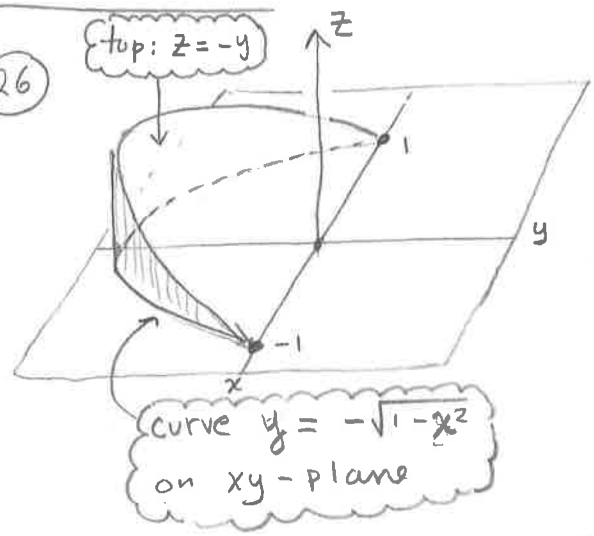

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} 1 \, dz \, dy \, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \left[z \right]_{0}^{-y} dy \, dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} -y \, dy \, dx$$

$$= \int_{-1}^{1} \left[-\frac{y^2}{2} \right]_{-\sqrt{1-x^2}}^{0} dx = \int_{-1}^{1} \frac{1-x^2}{2} \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1-x^2) \, dx = \frac{1}{2} \left[x - \frac{x^3}{3} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\left(1 - \frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right) \right] = \frac{1}{2} \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right)$$

$$= 1 - \frac{1}{3} = \frac{3}{3} - \frac{1}{3} = \boxed{\frac{2}{3} \text{ cubic units}}$$