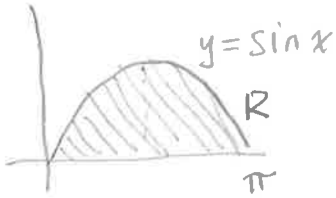
Section 15.6

(6) Find the centroid:



Let δ = density per square unit (constant)

$$\text{Mass} = M = \iint_R \delta \sin x \, dA = \delta \int_0^{\pi} \int_0^{\sin x} dy\, dx = \delta \int_0^{\pi} [y]_0^{\sin x} dx$$

$$= \delta \int_0^{\pi} \sin x \, dx = \delta \left[-\cos x\right]_0^{\pi} = \delta(-\cos\pi - (-\cos 0)) = \delta(1+1) = \boxed{2\delta}$$

$$M_x = \iint_R x\delta \, dA = \int_0^{\pi} [\delta x y]_0^{\sin x} dx = \int_0^{\pi} \delta x \sin x \, dx$$

$\int x \sin x \, dx = -x\cos x + \sin x$ by integration by parts

$$= \delta \int_0^{\pi} x \sin x \, dx = \delta\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \delta\left((-\pi \cos\pi + \sin\pi) - (-0\cos 0 + \sin 0)\right) = \delta(\pi - 0) = \boxed{\delta\pi}$$

$$M_y = \iint_R y\delta \, dA = \int_0^{\pi}\int_0^{\sin x} \delta y \, dy\, dx = \delta \int_0^{\pi} \left[\frac{y^2}{2}\right]_0^{\sin x} dx$$

$$= \delta \int_0^{\pi} \frac{\sin^2 x}{2} dx = \frac{\delta}{2}\int_0^{\pi} \sin^2 x \, dx = \frac{\delta}{2}\int_0^{\pi} \frac{1-\cos 2x}{2} dx$$

$$= \frac{\delta}{4}\int_0^{\pi} (1-\cos 2x)\, dx = \frac{\delta}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{\delta}{4}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 2\cdot 0\right)\right]$$

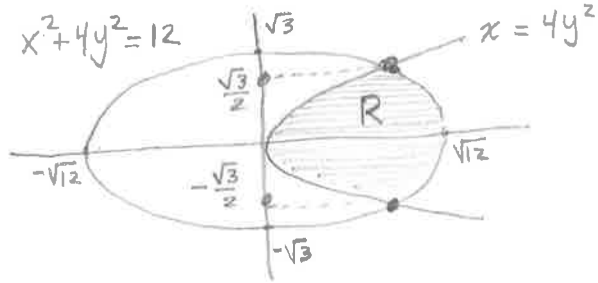
$$= \frac{\delta}{4}(\pi) = \boxed{\frac{\delta\pi}{4}}$$

$$\text{Centroid} = (\bar{x}, \bar{y}) = \left(\frac{M_x}{M}, \frac{M_y}{M}\right) = \left(\frac{\delta\pi}{2\delta}, \frac{\frac{\delta\pi}{4}}{2\delta}\right)$$

$$= \boxed{\left(\frac{\pi}{2}, \frac{\pi}{8}\right)}$$

Section 15.6

(12) Find the mass of this region if the density at (x,y) is $\delta(x,y) = 5x$



First let's find the points of intersection.

$$\left.\begin{array}{r} x^2+4y^2 = 12 \\ x = 4y^2 \end{array}\right\} (4y^2)^2 + 4y^2 = 12$$

$$16y^4 + 4y^2 - 12 = 0$$

$$4(4y^4 + y^2 - 3) = 0$$

$$4(4y^2-3)(y^2+1) = 0$$

$$\downarrow$$

$$4y^2 = 3$$

$$y = \pm\frac{\sqrt{3}}{2}$$

Therefore region lies between $y = \frac{\sqrt{3}}{2}$ and $y = -\frac{\sqrt{3}}{2}$

$$\text{Mass} = \iint_R \delta(x,y)\, dA = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_{4y^4}^{\sqrt{12-4y^2}} 5x \, dx\, dy$$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left[\frac{5x^2}{2}\right]_{4y^2}^{\sqrt{12-4y^2}} dx = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{5\sqrt{12-4y^2}^{\,2}}{2} - \frac{5(4y^2)^2}{2}\, dy$$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{5(12-4y^2) - 5(16y^4)}{2}\, dy = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} 30 - 10y^2 - 40y^4\, dy$$

$$= \left[30y - \frac{10}{3}y^3 - 8y^5\right]_{-\sqrt{3}/2}^{\sqrt{3}/2}$$

$$= \left(30\frac{\sqrt{3}}{2} - \frac{10}{3}\frac{3\sqrt{3}}{8} - 8\frac{9\sqrt{3}}{32}\right) - \left(30\left(-\frac{\sqrt{3}}{2}\right) - \frac{10}{3}\left(\frac{-3\sqrt{3}}{8}\right) - 8\left(-\frac{9\sqrt{3}}{32}\right)\right)$$

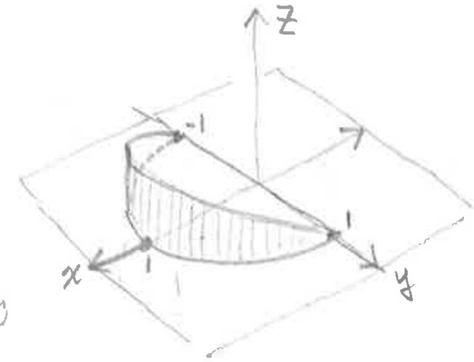
$$= 15\sqrt{3} - \frac{5}{4}\sqrt{3} - \frac{9}{4}\sqrt{3} + 15\sqrt{3} - \frac{5}{4}\sqrt{3} - \frac{9}{4}\sqrt{3}$$

$$= 30\sqrt{3} - \frac{5}{2}\sqrt{3} - \frac{9}{2}\sqrt{3} = \frac{60}{2}\sqrt{3} - \frac{5}{2}\sqrt{3} - \frac{9}{2}\sqrt{3}$$

$$= \frac{46}{2}\sqrt{3} = 23\sqrt{3}$$

Section 15.7

(14) $$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2+y^2)\,dz\,dx\,dy$$



By looking at the limits of integration we see that the region D lies over a half-circle (above x-axis) and under the plane $z = x$, as illustrated

In cylindrical coordinates, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. $0 \le r \le 1$ and $0 < z < x$
$r\cos\theta$

Thus the integral translates as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_0^{r\cos\theta}\left((r\cos\theta)^2+(r\sin\theta)^2\right)dz\,r\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_0^{r\cos\theta} r^2(\cos^2\theta+\sin^2\theta)\,r\,dz\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1\int_0^{r\cos\theta} r^3\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^1\left[r^3 z\right]_0^{r\cos\theta} dr\,d\theta$$

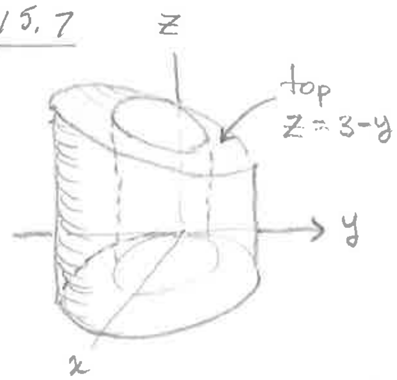
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^4\cos\theta\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^5}{5}\cos\theta\right]_0^1 d\theta$$

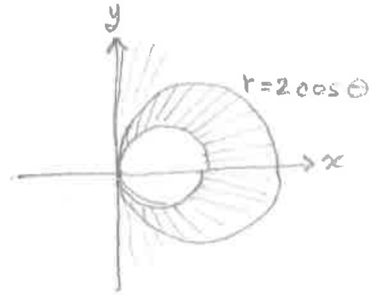
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{5}\cos\theta\,d\theta = \left[\frac{1}{5}\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{1}{5}\sin\frac{\pi}{2}-\frac{1}{5}\sin-\frac{\pi}{2} = \frac{1}{5}+\frac{1}{5} = \boxed{\frac{2}{5}}$$

Section 15.7

(18)





Note:

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

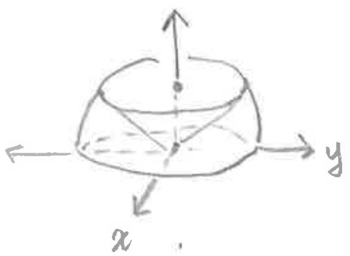
$\cos\theta \le r \le 2\cos\theta$

Also, the top is

$z = 3-y = 3 - r\sin\theta$

Answer $\displaystyle \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\cos\theta}^{2\cos\theta} \int_{0}^{3-r\sin\theta} f(r,\theta z)\, r \, dz\, dr\, d\theta$

(38)



$$V = \iiint_D dV = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2} \rho^2 \sin\phi \, dr\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^3}{3}\sin\phi\right]_0^2 d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \frac{8}{3}\sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{8}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi} -\frac{8}{3}\cos\frac{\pi}{3} + \frac{8}{3}\cos 0 \;\; d\theta$$

$$= \int_0^{2\pi}\left(-\frac{8}{3}\frac{1}{2}+\frac{8}{3}\right)d\theta = \int_0^{2\pi}\frac{4}{3}\, d\theta = \left[\frac{4}{3}\theta\right]_0^{2\pi}$$

$$= \boxed{\frac{8\pi}{3} \text{ cubic units}}$$