16.2

(10) a,b $C_1: \quad \vec{r}(t) = \langle t, t, t \rangle \quad 0 \le t \le 1$

$C_2: \quad \vec{r}(t) = \langle t, t^2, t^4 \rangle \quad 0 \le t \le 1$

$$\vec{F} = \langle xy, yz, xz \rangle$$

(a) $$\int_{C_1} F \cdot d\vec{r} = \int_0^1 F \cdot \frac{dr}{dt}\, dt = \int_0^1 \langle tt, tt, tt \rangle \cdot \langle 1,1,1 \rangle\, dt$$

$$= \int_0^1 3t^2\, dt = \left[t^3 \right]_0^1 = 1$$

(b) $$\int_{C_2} F \cdot d\vec{r} = \int_0^1 F \cdot \frac{dr}{dt}\, dt = \int_0^1 \langle tt^2, t^2t^4, tt^4 \rangle \cdot \langle 1, 2t, 4t^3 \rangle\, dt$$

$$= \int_0^1 \langle t^3, t^6, t^5 \rangle \cdot \langle 1, 2t, 4t^3 \rangle\, dt$$

$$= \int_0^1 (t^3 + 2t^7 + 4t^8)\, dt = \int_0^1 (t^3 + 2t^7 + 4t^8)\, dt$$

$$= \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4t^9}{9} \right] = \frac{1}{4} + \frac{1}{4} + \frac{4}{9} = \frac{1}{2} + \frac{4}{9} = \frac{9}{18} + \frac{8}{18} = \boxed{\frac{17}{18}}$$

(14) $\vec{r}(t) = \langle t, t^2 \rangle, \quad 1 \le t\ 2$

$$\int_C \frac{x}{y}\, dy = \int_1^2 \frac{t}{t^2}\, 2t\, dt = \int_1^2 2\, dt = \left[2t \right]_1^2 = 4 - 2 = \boxed{2}$$

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20 $F = \langle 2y, 3x, x+y \rangle \quad \vec{r}(t) = \langle \cos t, \sin t, \frac{t}{6} \rangle, \quad 0 \le t \le 2\pi.$

Find the work done by F over the curve $\vec{r}(t)$.

$$W = \int_C F \cdot T \, ds$$

$$= \int_0^{2\pi} F \cdot \frac{\vec{v}(t)}{|\vec{v}(t)|} |\vec{v}(t)| \, dt$$

$$= \int_0^{2\pi} F \cdot \vec{v}(t) \, dt$$

$$= \int_0^{2\pi} \langle 2\sin t, \ 3\cos t, \ \cos t + \sin t \rangle \cdot \langle -\sin t, \cos t, \frac{1}{6} \rangle dt$$

$$= \int_0^{2\pi} \left(-2\sin^2 t + 3\cos^2 t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt$$

$$= \int_0^{2\pi} \left(-2\,\frac{1-\cos 2t}{2} + 3\,\frac{1+\cos 2t}{2} + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt$$

$$= \int_0^{2\pi} \left(-1 + \cos 2t + \frac{3}{2} + \frac{3}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{5}{2}\cos 2t + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right) dt$$

$$= \left[\frac{1}{2}t + \frac{5}{4}\sin 2t + \frac{1}{6}\sin t - \frac{1}{6}\cos t\right]_0^{2\pi}$$

$$= \left(\frac{1}{2}2\pi + \frac{5}{4}\sin 4\pi + \frac{1}{6}\sin 2\pi - \frac{1}{6}\cos 2\pi\right) - \left(0 + \frac{5}{4}\sin 0 + \frac{1}{6}\sin 0 - \frac{1}{6}\cos 0\right)$$

$$= \left(\pi + 0 + 0 - \frac{1}{6}\right) - \left(0 + 0 + 0 - \frac{1}{6}\right) = \boxed{\pi}$$

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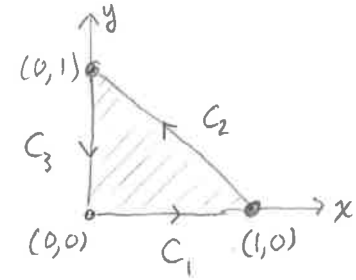
(24)



$C_1 : \vec{r}(t) = \langle t, 0 \rangle, \quad 0 \le t \le 1$

$C_2 : \vec{r}(t) = \langle 1-t, t \rangle, \quad 0 \le t \le 1.$

$C_3 : \vec{r}(t) = \langle 0, 1-t \rangle, \quad 0 \le t \le 1$

Evaluate $\int_C (x-y)\,dx + (x+y)\,dy$

Breaking up C into parts C_1, C_2, C_3, the answer will be the sum of the following integrals:

$$\int_{C_1} (x-y)\,dx = \int_0^1 (t-0)\frac{dx}{dt}\,dt = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\int_{C_1} (x+y)\,dy = \int_0^1 (t+0)\frac{dy}{dt}\,dt = \int_0^1 t \cdot 0\,dt = 0$$

$$\int_{C_2} (x-y)\,dx = \int_0^1 (1-t-t)\frac{dx}{dt}\,dt = \int_0^1 (1-2t)(-1)\,dt = \int_0^1 (2t-1)\,dt = \left[t^2-t\right]_0^1 = 0$$

$$\int_{C_2} (x+y)\,dy = \int_0^1 (1-t+t)\frac{dy}{dt}\,dt = \int_0^1 1 \cdot 1\,dt = \int_0^1 dt = \left[t\right]_0^1 = 1$$

$$\int_{C_3} (x-y)\,dx = \int_0^1 (0-(1-t))\frac{dx}{dt}\,dt = \int_0^1 (t-1)\cdot 0\,dt = 0$$

$$\int_{C_3} (x+y)\,dy = \int_0^1 (0+(1-t))\frac{dy}{dt}\,dt = \int_0^1 (1-t)(-1)\,dt = \int_0^1 (t-1)\,dt = \left[\frac{t^2}{2}-t\right]_0^1$$

$$= -\frac{1}{2}$$

From the above, $\int_C (x-y)\,dx + (x+y)\,dy =$

$$\frac{1}{2} + 0 + 0 + 1 + 0 - \frac{1}{2} = \boxed{1}$$

16.3 ⑥ Is $\vec{F}(x,y,z) = \langle e^x \cos y, -e^x \sin y, z \rangle$ conservative?

We apply the component test for conservative fields. This involves the following computations:

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[z] = 0 \qquad \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}\left[e^x \sin y\right] = 0$$

$$\frac{\partial M}{\partial z} = \frac{\partial}{\partial z}\left[e^x \cos y\right] = 0 \qquad \frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left[z\right] = 0$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[-e^x \sin y\right] = -e^x \sin y \qquad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[e^x \cos y\right] = -e^x \sin y$$

Since $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$, $\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$, the answer is $\boxed{\text{YES}}$ the field is conservative

⑩ Find a potential function for $\vec{F} = \langle y \sin z, x \sin z, xy \cos z \rangle$

We seek a function f with $\nabla f = \langle \underbrace{y \sin z}_{\frac{\partial f}{\partial x}}, \underbrace{x \sin z}_{\frac{\partial f}{\partial y}}, \underbrace{xy \cos z}_{\frac{\partial f}{\partial z}} \rangle$

Since $\frac{\partial f}{\partial x} = y \sin z$,

$$f(x,y,z) = \int y \sin z \, dx = xy \sin z + g(y,z)$$

Then $\frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z$

Thus $\frac{\partial g}{\partial y} = 0$ so $g(y,z) = h(z)$ depends only on z

Thus $f(x,y,z) = xy \sin z + h(z)$

so $\frac{\partial f}{\partial z} = xy \cos z + \frac{\partial h}{\partial z} = xy \cos z$

Thus $\frac{\partial h}{\partial z} = 0$ so $h(z) = C$ (constant)

Consequently $\boxed{f(x,y,z) = xy \sin z + C}$ is a potential function.

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(20) Evaluate this integral given that the associated field is conservtive:

$$\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$$

The answer will be $f(2,1,1) - f(1,2,1)$ where f satisfies

$\nabla f = \left\langle 2x\ln y - yz,\ \frac{x^2}{y} - xz,\ -xy \right\rangle$, or

(i) $\frac{\partial f}{\partial x} = 2x\ln y - yz$

(ii) $\frac{\partial f}{\partial y} = \frac{x^2}{y} - xz$

(iii) $\frac{\partial f}{\partial z} = -xy$

From (iii) $f(x,y,z) = \int -xy\,dz = -xyz + g(x,y)$ $\quad (*)$

Taking $\frac{\partial}{\partial y}$ of this and equating with (ii) yields

$$\frac{\partial}{\partial y}\left[f(x,y,z)\right] = -xz + \frac{\partial g}{\partial y} = \frac{x^2}{y} - xz$$

Therefore $\frac{\partial g}{\partial y} = \frac{x^2}{y} \Rightarrow g(x,y) = \int \frac{x^2}{y}\,dy = x^2\ln y + C \quad (**)$

Now combining $(*)$ and $(**)$ we get

$f(x,y,z) = x^2\ln y - xyz + C$

Let $C = 0$, and we get a potential function $f(x,y,z) = x^2\ln y - xyz$

Then

$$\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{2} - xz\right)dx - xy\,dz$$

$= f(2,1,1) - f(1,2,1) = \left(2^2\ln 1 - 2\cdot 1\cdot 1\right) - \left(1^2\ln 2\cdot 1 - 1\cdot 2\cdot 1\right)$

$= (4\cdot 0 - 2) - (\ln 2 - 2) = -\ln 2 = \boxed{\ln\frac{1}{2}}$