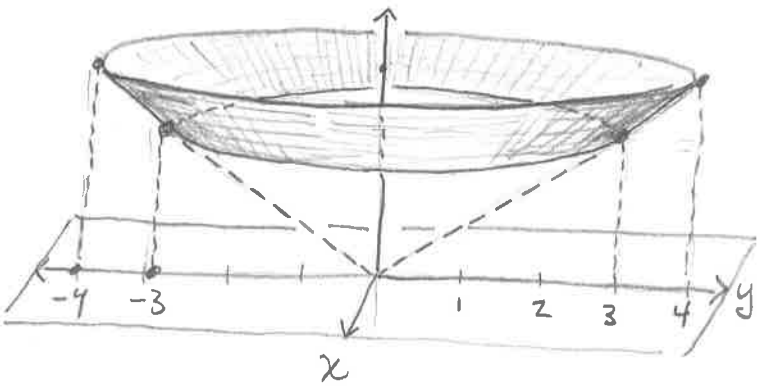
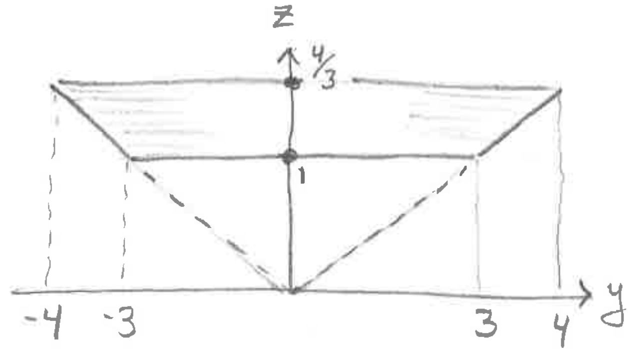
16.5

(20) Find the surface area of the portion of the cone $z = \frac{1}{3}\sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = \frac{4}{3}$.



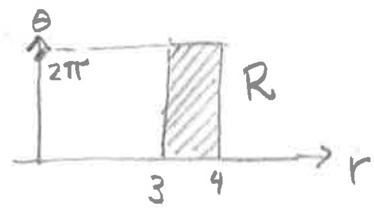

SIDE VIEW

Parameterization:

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $z = \frac{1}{3}\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \frac{r}{3}$

Thus the surface is described parametrically by

$\vec{r}(r,\theta) = \left\langle r\cos\theta,\ r\sin\theta,\ \frac{r}{3} \right\rangle$, $0 \le \theta \le 2\pi$ and $3 \le r \le 4$.




Then: $\vec{r}_r = \left\langle \cos\theta,\ \sin\theta,\ \frac{1}{3} \right\rangle$

$\vec{r}_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 0 \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left\langle -\frac{r}{3}\cos\theta,\ \frac{r}{3}\sin\theta,\ r\cos^2\theta + r\sin^2\theta \right\rangle$$

$$= \left\langle -\frac{r}{3}\cos\theta,\ \frac{r}{3}\sin\theta,\ r \right\rangle$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{\left(\frac{-r}{3}\cos\theta\right)^2 + \left(\frac{r}{3}\sin\theta\right)^2 + r^2} = r\sqrt{\frac{1}{9} + 1} = r\frac{\sqrt{10}}{3}$$

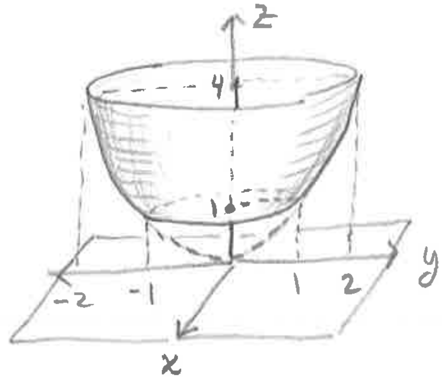
$$\text{Area} = \iint_R |\vec{r}_r \times \vec{r}_\theta|\, dA = \int_0^{2\pi}\int_3^4 r\frac{\sqrt{10}}{3}\, dr\, d\theta = \int_0^{2\pi} \left[ r^2 \frac{\sqrt{10}}{6} \right]_3^4 d\theta$$

$$= \int_0^{2\pi} \left( 16\frac{\sqrt{10}}{6} - 9\frac{\sqrt{10}}{6} \right) d\theta = \frac{7}{6}\sqrt{10}\int_0^{2\pi} d\theta = \frac{7}{6}\sqrt{10}\,\left[\theta\right]_0^{2\pi}$$

$$= \boxed{\frac{7\pi\sqrt{10}}{3} \text{ square units}}$$

(24) Find the surface area of the paraboloid $z = x^2 + y^2$ between $z = 1$ and $z = 4$.

First draw the picture




For the parameterization, let $x = u\cos v$ and $y = u\sin v$ so that $z = (u\cos v)^2 + (u\sin v)^2 = u^2$. Thus the surface has the following parametric description:

$$\vec{r}(u,v) = \left\langle u\cos v,\ u\sin v,\ u^2 \right\rangle,\quad 1 \le u \le 2,\quad 0 \le v \le 2\pi.$$

$$\vec{r}_u = \langle \cos v,\ \sin v,\ 2u \rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v\ \ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \left\langle -2u^2\cos v,\ -2u^2\sin v,\ u\cos^2 v + u\sin^2 v \right\rangle$$

$$= \left\langle -2u^2\cos v,\ -2u^2\sin v,\ u \right\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-2u^2\cos v)^2 + (-2u^2\sin v)^2 + u^2} = \sqrt{4u^4 + u^2} = \sqrt{4u^2+1}\ u$$

$$\text{Area} = \int_0^{2\pi}\int_1^2 \frac{1}{8}\sqrt{4u^2+1}\ 8\,du\,dv$$

$w = 4u^2 + 1$
$dw = 8u\,dw$

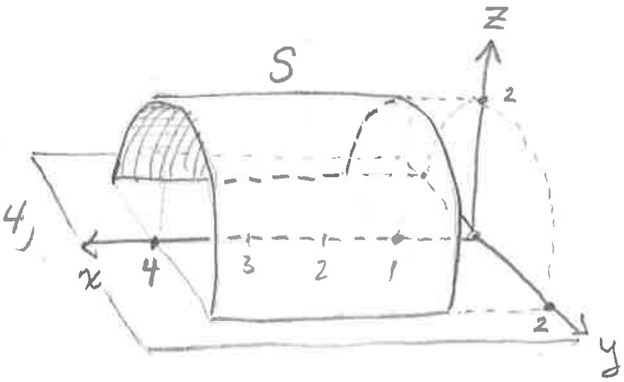
$$= \int_0^{2\pi}\int_{4\cdot 1^2+1}^{4\cdot 2^2+1} \frac{1}{8}\sqrt{w}\,dw\,dv = \int_0^{2\pi}\left[\frac{1}{8}\frac{2}{3}\sqrt{w}^{\,3}\right]_5^{17} dv$$

$$\int_0^{2\pi}\left(\frac{\sqrt{17}^{\,3}}{12} - \frac{\sqrt{5}^{\,3}}{12}\right)dv = \frac{17\sqrt{17} - 5\sqrt{5}}{12}\int_0^{2\pi} dv$$

$$= \frac{17\sqrt{17} - 5\sqrt{5}}{12}\,2\pi = \boxed{\frac{17\sqrt{17} - 5\sqrt{5}}{6}\,\pi \text{ square units}}$$

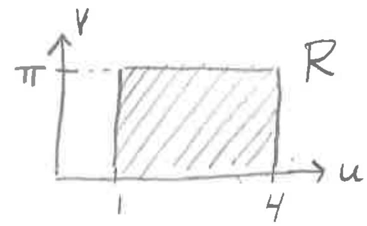
16.6

② Integrate $G(x,y,z)=z$ over the cylindrical surface $y^2+z^2=4$, $z\geq 0$, $1\leq x\leq 4$.




This surface can be parameterized as follows:

$\vec{r}(u,v)=\langle u,\ 2\cos v,\ 2\sin v\rangle$, $1\leq u\leq 4$, $0\leq v\leq \pi$.




$\vec{r}_u=\langle 1,0,0\rangle$

$\vec{r}_v=\langle 0,-2\sin v,\ 2\cos v\rangle$

$$\vec{r}_u\times\vec{r}_v=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 0\\ 0 & -2\sin v & 2\cos v\end{vmatrix}=\langle 0,-2\cos v,-2\sin v\rangle$$

$$|\vec{r}_u\times\vec{r}_v|=\sqrt{0^2+(-2\cos v)^2+(-2\sin v)^2}=\sqrt{4}=2$$

Therefore $\displaystyle\iint_S G(x,y,z)\,d\sigma=\iint_R G(u,2\cos v,2\sin v)\,|\vec{r}_u\times\vec{r}_v|\,dA$

$$=\int_0^{\pi}\int_1^4 2\sin v\cdot 2\,du\,dv=\int_0^{\pi}\int_1^4 4\sin v\,du\,dv$$

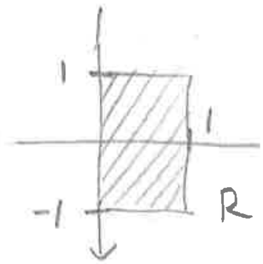
$$=\int_0^{\pi}\Big[4u\sin v\Big]_1^4\,dv=\int_0^{\pi}12\sin v\,dv$$

$$=12\int_0^{\pi}\sin v\,dv=12\Big[-\cos v\Big]_0^{\pi}=12\big(-(-1)-(-1)\big)$$

$$=\boxed{24}$$

16.6

(16) Integrate $G(x,y,z) = x$ over the surface $z = x^2 + y$, $0 \le x \le 1$, $-1 \le y \le 1$




This surface is described explicitly over the rectangle $R: \ 0 \le x \le 1, \ -1 \le y \le 1$, i.e. it is the graph of the function $z = f(x,y) = x^2 + y$

$$\iint_S G(x,y,z)\,d\sigma = \iint_R G(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\,dx\,dy$$

$$= \int_{-1}^{1}\int_0^1 x\sqrt{(2x)^2 + 1^2 + 1}\,dx\,dy$$

$u = 4x^2 + 2$

$du = 8x\,dx$

$$= \int_{-1}^{1}\int_0^1 \sqrt{4x^2+2}\;x\,dx\,dy$$

$$= \int_{-1}^{1}\int_0^1 \frac{1}{8}\sqrt{4x^2+2}\;8x\,dx\,dy$$

$$= \int_{-1}^{1}\int_{4\cdot 0^2+2}^{4\cdot 1^2+2} \frac{1}{8}\sqrt{u}\,du\,dy = \int_{-1}^{1}\left[\frac{1}{8}\frac{2}{3}\sqrt{u}^3\right]_2^6 dy$$

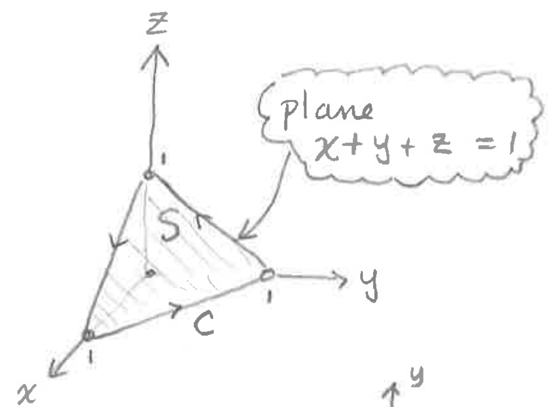
$$= \int_{-1}^{1}\left(\frac{1}{12}\sqrt{6}^3 - \frac{1}{12}\sqrt{2}^3\right)dy = \frac{6\sqrt{6} - 2\sqrt{2}}{12}\int_{-1}^{1} dy = \frac{3\sqrt{6} - \sqrt{2}}{6}\Big[y\Big]_{-1}^{1}$$

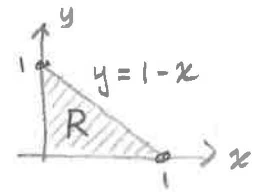
$$= \boxed{\frac{3\sqrt{6} - \sqrt{2}}{3}}$$

16.7 (4)

$F = \langle y^2+z^2,\ x^2+z^2,\ x^2+y^2 \rangle$

Find the circulation of $F$ around the curve $C$, here




The plane is $z = 1 - x - y$ defined on this region:

Unit normal to plane is $\vec{n} = \dfrac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$ (Chapter 12 ! )

Also $\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & x^2+z^2 & x^2+y^2 \end{vmatrix} = \langle 2y-2z,\ 2z-2x,\ 2x-2y \rangle$

So $\nabla \times f \cdot \vec{n} = \langle 2y-2z,\ 2z-2x,\ 2x-2y\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

$$= \frac{2y-2z}{\sqrt{3}} + \frac{2z-2x}{\sqrt{3}} + \frac{2x-2y}{\sqrt{3}} = 0$$

Consequently, the circulation is

$$\oint_C F \cdot dr = \iint_S \nabla \times f \cdot \vec{n}\, d\sigma = \iint_S 0\, d\sigma = \boxed{0}$$