The algebra of vectors

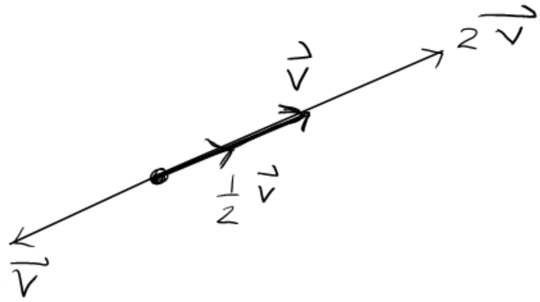
Vectors can be added, subtracted and multiplied in various ways. The algebraic system that results from these operations is very useful. We'll explore some of these operations today — more to come in 12.3 and 12.4.

Scalar Multiplication

A vector $\vec{v}$ can be multiplied by a number (or "scalar") to get a new vector $k\vec{v}$

$$k\langle v_1, v_2\rangle = \langle kv_1, kv_2\rangle$$

$$k\langle v_1, v_2, v_3\rangle = \langle kv_1, kv_2, kv_3\rangle$$



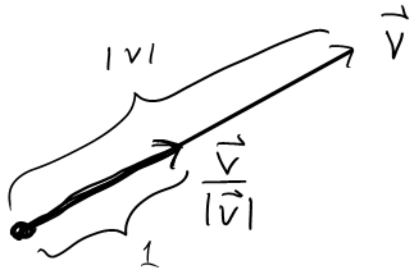
Thus $k\vec{v}$ is just $\vec{v}$ scaled by a factor of k
Vector $k\vec{v}$ is called a "scalar multiple of $\vec{v}$.

Ex $-5\langle 3, -2, 0\rangle = \langle -15, 10, 0\rangle$ $\qquad$ $0\langle 10, 5\rangle = \langle 0, 0\rangle$

Easy to confirm: $\boxed{|k\vec{v}| = |k|\cdot|\vec{v}|}$

absolute value $\quad$ norm

Often, given a vector $\vec{v}$ you'll need to compute a unit vector (i.e. one with length 1) having the same direction as $\vec{v}$.
This vector is the scalar multiple $\frac{1}{|\vec{v}|}\vec{v} = \frac{\vec{v}}{|\vec{v}|}$



Conclusion

① Unit vector in the direction of $\vec{v}$ is $\frac{1}{|\vec{v}|}\vec{v} = \frac{\vec{v}}{|\vec{v}|}$

② $\vec{v} = |\vec{v}|\frac{\vec{v}}{|\vec{v}|}$ ⟵ expresses $\vec{v}$ as its length times direction.

Ex Find unit vector in the direction of $\vec{v} = \langle 2, -1, 1\rangle$

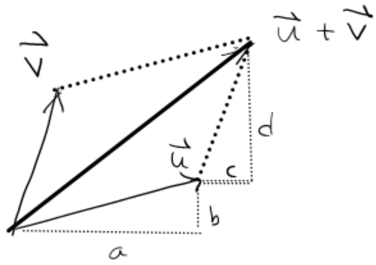
$|\vec{v}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$ so answer is $\frac{1}{\sqrt{6}}\langle 2, -1, 1\rangle = \left\langle \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\rangle$

Vector Addition

Vectors $\vec{u}$ and $\vec{v}$ can be added to get a vector $\vec{u}+\vec{v}$.
This is best described with vectors in standard position

2-D: $\langle a, b \rangle + \langle c, d \rangle = \langle a+c, b+d \rangle$

3-D: $\langle a, b, c \rangle + \langle d, e, f \rangle = \langle a+d, b+e, c+f \rangle$



Geometrically $\vec{u}+\vec{v}$ is the diagonal of the parallelogram formed by $\vec{u}$ and $\vec{v}$. This is true for vectors in 2-D as well as in 3-D

Example: $\langle 5, -1, 3 \rangle + \langle \frac{1}{2}, 1, 2 \rangle = \langle \frac{11}{2}, 0, 5 \rangle$.

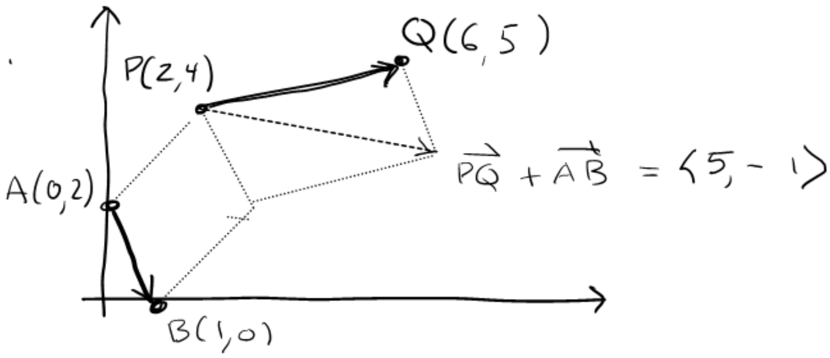
Example Find $\overrightarrow{PQ} + \overrightarrow{AB}$

$\overrightarrow{PQ} = \langle 6-2, 5-4 \rangle = \langle 4, 1 \rangle$

$\overrightarrow{AB} = \langle 1, -2 \rangle$

$\overrightarrow{PQ} + \overrightarrow{AB} = \langle 4, 1 \rangle + \langle 1, -2 \rangle$

$= \langle 5, -1 \rangle$

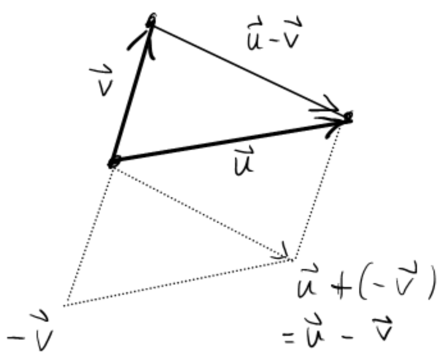


Vector Subtraction

Vectors $\vec{u}$ and $\vec{v}$ can be subtracted to get a vector $\vec{u}-\vec{v}$.
Definition: $\vec{u}-\vec{v} = \vec{u} + (-1)\vec{v}$. Therefore:

2-D: $\langle a, b \rangle - \langle c, d \rangle = \langle a-c, b-d \rangle$

3-D: $\langle a, b, c \rangle - \langle d, e, f \rangle = \langle a-d, b-e, c-f \rangle$



Geometrically, $\vec{u}-\vec{v}$ is the vector directed from the tip of $\vec{v}$ to the tip of $\vec{u}$, as illustrated.

Alternatively note that $\vec{u}-\vec{v}$, as drawn, satisfies $(\vec{u}-\vec{v}) + \vec{v} = \vec{u}$

The zero vector is $\begin{cases} \vec{0} = \langle 0,0 \rangle & \text{in 2-D} \\ \vec{0} = \langle 0,0,0 \rangle & \text{in 3-D} \end{cases}$

Notice that $\vec{u} - \vec{u} = \vec{0}$, as expected

Properties of vector operations

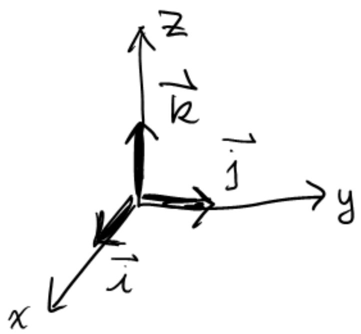
The following are easy to verify, where a, b are scalars (numbers)

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} - \vec{u} = \vec{0}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) = \vec{u} + \vec{v} + \vec{w}$
- $|a\vec{u}| = |a| \cdot |\vec{u}|$
- $0\vec{u} = \vec{0}$
- $1\vec{u} = \vec{u}$
- $(a+b)\vec{u} = a\vec{u} + b\vec{u}$
- $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- $a(bu) = (ab)\vec{u}$

We use these properties often, usually intuitively without giving it a second thought. However, we should not take them for granted. Soon we will introduce a vector product and it will happen that $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$. The point is that things don't always happen as we might expect. But the above properties indicate that the operations introduced today behave in predictable (and useful) ways.

Component Form

Recall $\begin{cases} \vec{i} = \langle 1,0,0 \rangle \\ \vec{j} = \langle 0,1,0 \rangle \\ \vec{k} = \langle 0,0,1 \rangle \end{cases}$



Note $\langle a, b, c \rangle = \underbrace{a\vec{i} + b\vec{j} + c\vec{k}}_{\text{"component form" of } \langle a,b,c \rangle}$

Sometimes it's convenient to write a vector in its component form. Read what the text has to say about this!