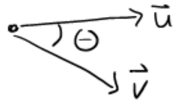
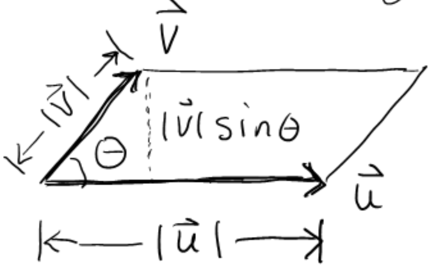
Recall • Dot Product $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$



• Determinant of a 2×2 matrix: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

• Determinant of a 3×3 matrix $\begin{vmatrix} a & b & c \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = a\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - b\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + c\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$

* Area of parallelogram formed by $\vec{u}$ and $\vec{v}$ is $A = |u||v|\sin\theta$

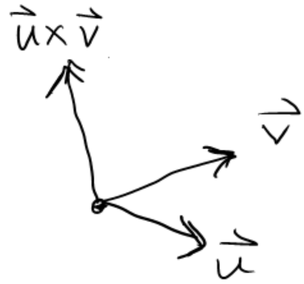


$|\vec{v}|\sin\theta$ (height)

$A = (\text{base})(\text{height})$

$= |\vec{u}|\cdot|\vec{v}|\sin\theta$

Section 12.4 The Cross Product



Goal Define a product × on vectors in $\mathbb{R}^3$ so $\vec{u}\times\vec{v} = $ (vector orthogonal to both $\vec{u}$ and $\vec{v}$).

Definition The cross product of $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ is

$$\vec{u}\times\vec{v} = \langle u_2v_3 - u_3v_2,\ -u_1v_3 + u_3v_1,\ u_1v_2 - u_2v_1\rangle \tag{1}$$

$$= \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{2}$$

Example $\langle 3, 2, 1\rangle \times \langle 1, 4, 2\rangle =$

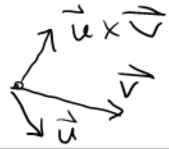
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ 1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} i - \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} j + \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}\vec{k} = 0\vec{i} - 5\vec{j} + 10\vec{k} = \langle 0, -5, 10\rangle$$

Note this is orthogonal to both $\langle 3, 2, 1\rangle$ and $\langle 1, 4, 2\rangle$

Easy to check from the definition (1) that: $\begin{cases} (\vec{u}\times\vec{v})\cdot\vec{u} = 0 \\ (\vec{u}\times\vec{v})\cdot\vec{v} = 0 \end{cases}$

Therefore:

$\boxed{\text{The vector } \vec{u}\times\vec{v} \text{ is orthogonal to both } \vec{u} \text{ and } \vec{v}}$



Also, from properties of determinants, part (2) gives.

Properties

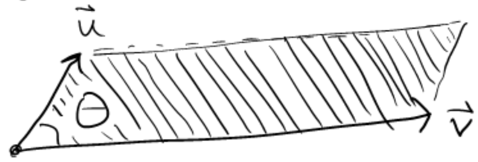
- $(r\vec{u})\times(s\vec{v}) = rs(\vec{u}\times\vec{v})$
- $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$ ← $\times$ is <u>not</u> commutative.
- $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times w$
- $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$

} distribututive laws

- $\vec{0}\times\vec{u} = \vec{0}$

Here is another fundamental property of $\times$.

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta = \begin{pmatrix}\text{area of parallelogram} \\ \text{spanned by } \vec{u} \text{ and } \vec{v}.\end{pmatrix}$$



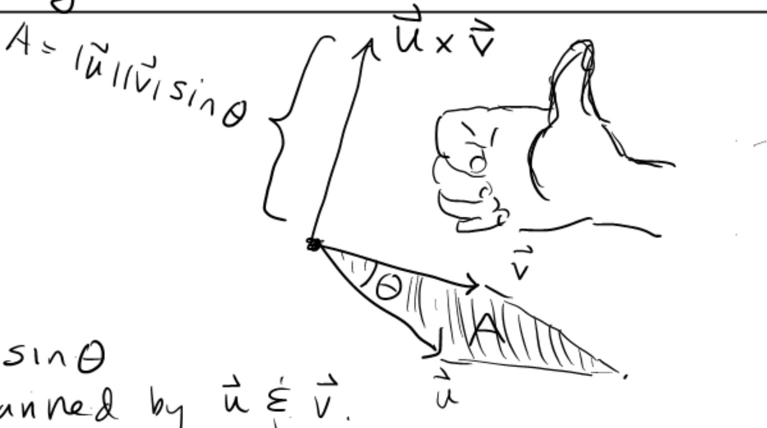
<u>Proof</u> $(|\vec{u}||\vec{v}|\sin\theta)^2 = |\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta) = |\vec{u}|^2|\vec{v}|^2 - |\vec{u}|^2|\vec{v}|^2\cos^2\theta$

$= \vec{u}\cdot\vec{u}\ \vec{v}\cdot\vec{v} - (\vec{u}\cdot\vec{v})^2 = \ldots$ keep going $\ldots$

$= |\vec{u}\times\vec{v}|^2$

From the above we get the following fundamental interpretation:

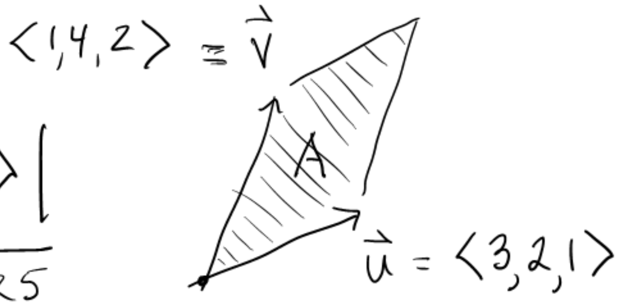
The vector $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$, pointing in the direction given by the right-hand rule.
Its magnitude is $|\vec{u}||\vec{v}|\sin\theta$
= area of parallelogram spanned by $\vec{u}$ & $\vec{v}$.



<u>Ex.</u>

$i\times j = k$	$j\times i = -k$	$i\times i = 0$
$j\times k = i$	$k\times j = -i$	$j\times j = 0$
$k\times i = j$	$i\times k = -j$	$k\times k = 0$

<u>Example</u> Find the area of this parallelogram in $\mathbb{R}^3$.

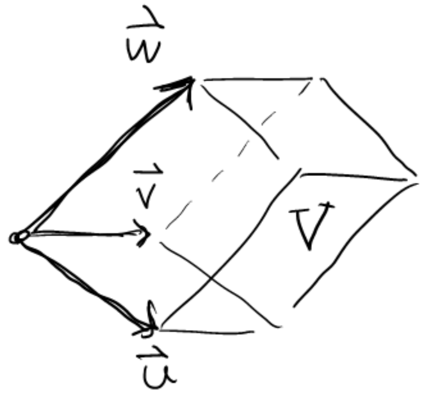


$$A = |\vec{u}\times\vec{v}| = |\langle 0, -5, 10\rangle|$$

$$= \sqrt{0^2 + (-5)^2 + 10^2} = \sqrt{125}$$

$$\approx \boxed{11.1803 \text{ square units}}$$

<u>Triple Scalar Product</u>



Text shows that the volume of the parallelepiped spanned by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is

$$V = (\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

It is possible that this could work out to be <u>negative</u>. Take the absolute value if you're looking for volume.