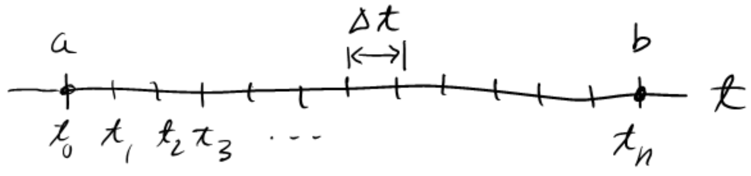
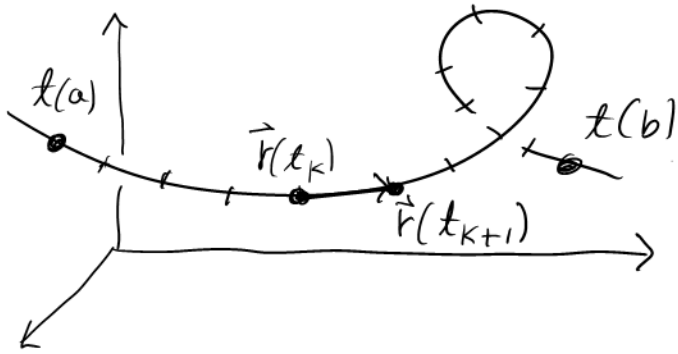
Section 13.3 Arc Length



Problem Find length of curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ between $t = a$ and $t = b$



$$L \approx \sum_{k=0}^{n-1} \left| \vec{r}(t_{k+1}) - \vec{r}(t_k) \right|$$

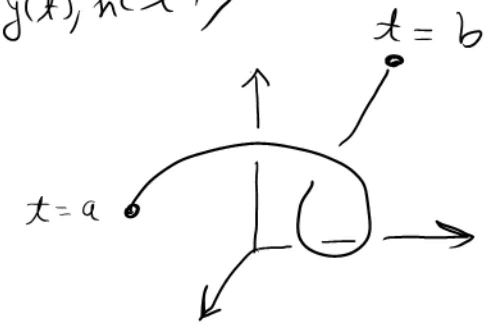
$$= \sum_{k=0}^{n-1} \left| \frac{\vec{r}(t_k + \Delta t) - r(t_k)}{\Delta t} \right| \Delta t$$

$$\simeq \sum_{k=0}^{n-1} \left| \vec{r}\,'(t_k) \right| \Delta t$$

$$L = \lim_{n \to \infty} \sum_{k=0}^{n} \left| \vec{r}\,'(t_k) \right| \Delta t = \int_a^b |\vec{r}\,'(t)|\, dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\, dt$$

Conclusion

The arc length of the curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ between $t = a$ and $t = b$ is



$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

$$= \int_a^b |\vec{r}\,'(t)|\, dt$$

Example Find the length of $\vec{r}(t) = \left\langle \sqrt{6}t^2, \frac{2t^3}{3}, 6t \right\rangle$ between $t = 3$ and $t = 6$

$$L = \int_3^6 \sqrt{(2\sqrt{6}\,t)^2 + (2t^2)^2 + 6^2}\, dt = \int_3^6 \sqrt{24t^2 + 4t^4 + 36}\, dt$$

$$= \int_3^6 2\sqrt{t^4 + 6t^2 + 9}\, dt = \int_3^6 2\sqrt{(t^2 + 3)^2}\, dt$$

$$= 2\int_3^6 (t^2 + 3)\, dt = 2\left[\frac{t^3}{3} + 3t\right]_3^6 = 2\left[\left(\frac{6^3}{3} + 3\cdot 6\right) - \left(\frac{3^3}{3} + 3\cdot 3\right)\right]$$

$$= 2\left[(72 + 18) - (9 + 9)\right] = \boxed{144 \text{ units}}$$

Consider a moving object whose position at time t is $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$.

From time $t=a$ to time t object has moved along the curve a distance of

$$s(t) = \int_a^t \sqrt{f'(\tau)^2 + g'(\tau)^2 + h'(\tau)^2}\, d\tau = \int_a^t |v(\tau)|\, d\tau$$

By Fundamental Theorem of Calculus,

$$s'(t) = \frac{d}{dt}\left[\int_a^t |v(\tau)|\, d\tau \right] = |v(t)| = \left(\begin{array}{c} \text{speed at} \\ \text{time } t \end{array} \right)$$

Notice that this says

$$\left(\begin{array}{c} \text{rate of change} \\ \text{of dist. traveled} \\ \text{at time } t \end{array} \right) = \left(\begin{array}{c} \text{speed at} \\ \text{time } t \end{array} \right)$$

which is <u>exactly</u> what you would expect!