Chapter 14 Partial Derivatives

Types of functions

$y = f(x)$ .... $f: \mathbb{R} \to \mathbb{R}$ } Calc I, II

$\langle f(t), g(t), h(t) \rangle = \vec{r}(t)$ .... $\vec{r}: \mathbb{R} \to \mathbb{R}^3$ } chapter 13

$\langle f(t), g(t) \rangle = \vec{r}(t)$ ..... $\vec{r}: \mathbb{R} \to \mathbb{R}^2$

$z = f(x, y)$ .... $f: \mathbb{R}^2 \to \mathbb{R}$ } Chapter 14

So we will begin to concern ourselves with functions of the form $f(x, y)$, and their derivatives. But before doing any calculus, we explore this kind of function

Section 14.1 Functions of Several Variables

Here are examples of functions of several variables

$\underline{f(x, y) = x^2 y + 2x}$

$f(2, 3) = 2^2 \cdot 3 + 2 \cdot 2 = 16$

$f(0, 1) = 0^2 \cdot 1 + 2 \cdot 0 = 0$

$f(1, 0) = 1^2 \cdot 0 + 2 \cdot 1 = 2$

$\underline{g(x, y, z) = \sqrt{x^2 + y^2 + z^2}}$

$g(3, 0, 2) = \sqrt{3^2 + 0^2 + 2^2} = \sqrt{13}$

$g(-1, -1, -1) = \sqrt{3}$

In such functions you plug in an ordered pair or triple and get a single number as output.

Could even have a function of n variables

$$\underset{\text{dependent variable}}{z} = f(\underbrace{x_1, x_2, x_3, \dots, x_n}_{\text{independent variables}})$$

Domain All n-tuples $(x_1, x_2, \dots, x_n)$ for which f is defined or meaningful

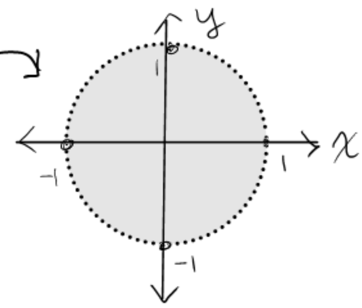
Range Set of all possible output values z.

Example $g(x,y) = \dfrac{1}{\sqrt{1-x^2-y^2}}$

Find domain Must have $1-x^2-y^2>0$

$x^2+y^2<1$

Domain is all (x,y) inside unit circle



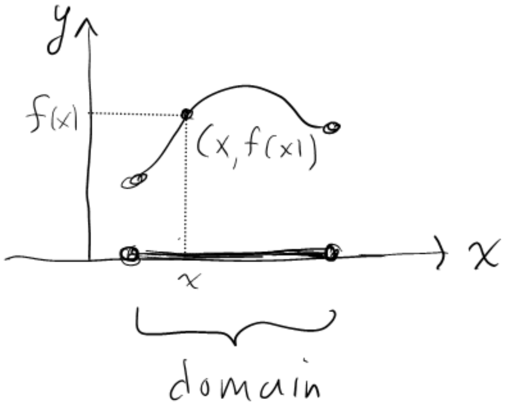
Find range Note: $g(x,y) = \dfrac{1}{\sqrt{1-x^2-y^2}} \geq 1$

$g(x,0) = \dfrac{1}{\sqrt{1-x^2}}$ can take on any value $z \geq 1$.

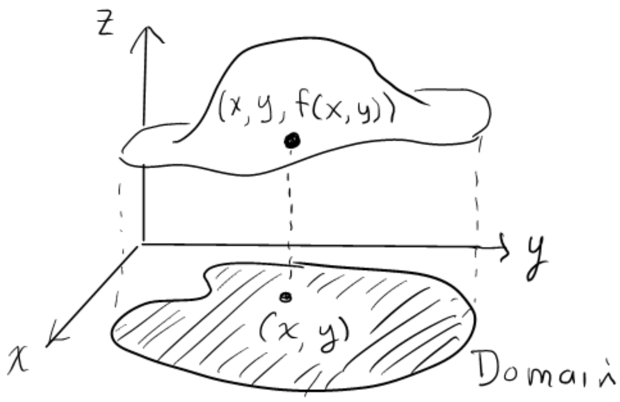
Range is $[1,\infty)$

GRAPHS

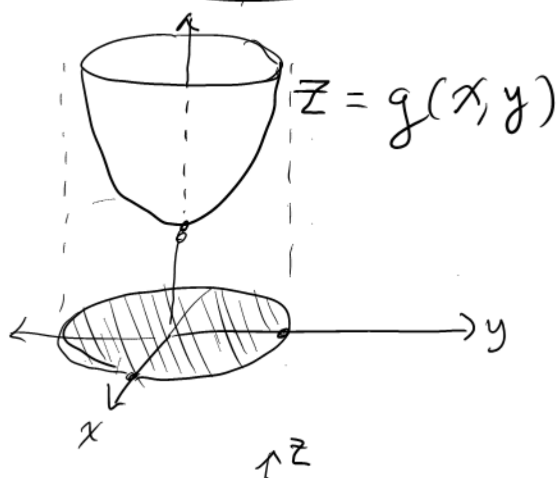
Recall: The graph of $y=f(x)$ is set of points $(x, f(x))$



The graph of $z=f(x,y)$ is set of points $(x,y,f(x,y))$



Example $g(x,y) = \dfrac{1}{\sqrt{1-x^2-y^2}}$

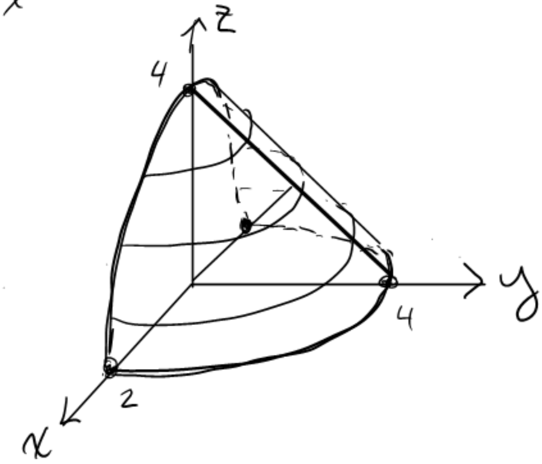


Example $f(x,y) = 4-x^2-y$

xz-plane $z = 4-x^2-0 \rightsquigarrow z = 4-x^2$

yz-plane $z = 4-0^2-y \rightsquigarrow z = 4-y$

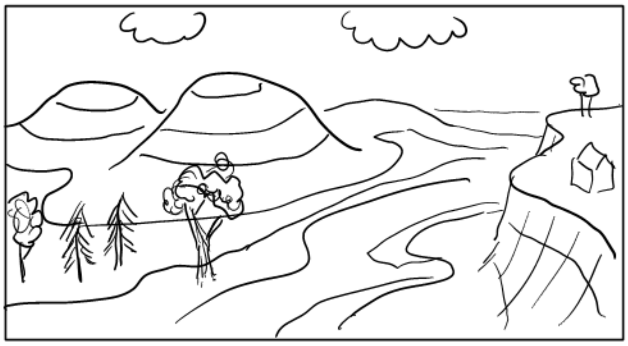
xy-plane $0 = 4-x^2-y \rightsquigarrow y = 4-x^2$



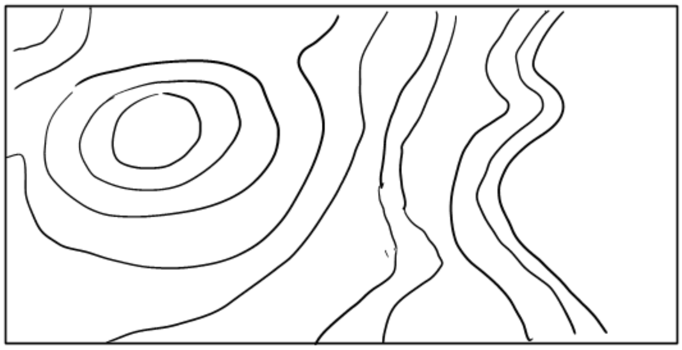
(Domain – xy-plane)

Level Curves

Here is another way to visualize a function $z = f(x,y)$ of **two** variables. It uses the same idea of a topographical map..

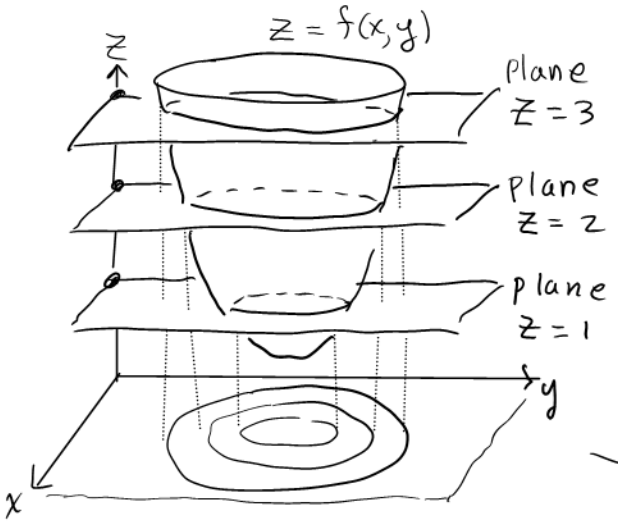


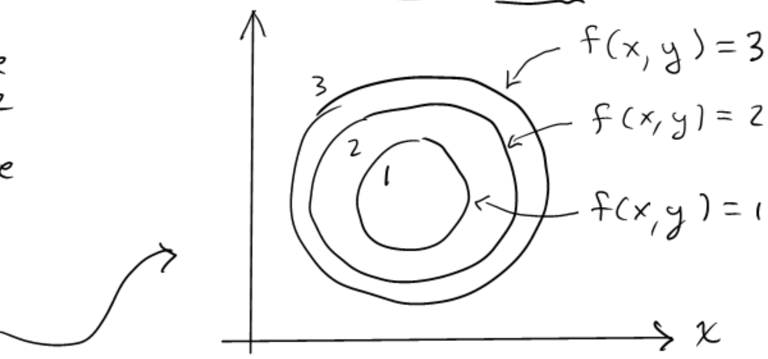
3-D world. - somewhat like a graph of $z = f(x,y)$



Topographic map with "level curves" indicating different elevations.

The same idea applies to graphs of $z = f(x,y)$. On the xy-plane the points (x,y) that give an elevation of $z = k$ are the graph of the equation $f(x,y) = k$. This is called the level curve for $z = k$.





Example Sketch the level curves for $f(x,y) = 4 - x^2 - y$

For level $z = k$, level curve is
$$f(x,y) = k$$
$$4 - x^2 - y = k$$
$$\boxed{y = 4 - k - x^2}$$

$z = 5$: $y = -1 - x^2$

$z = 4$ $y = -x^2$

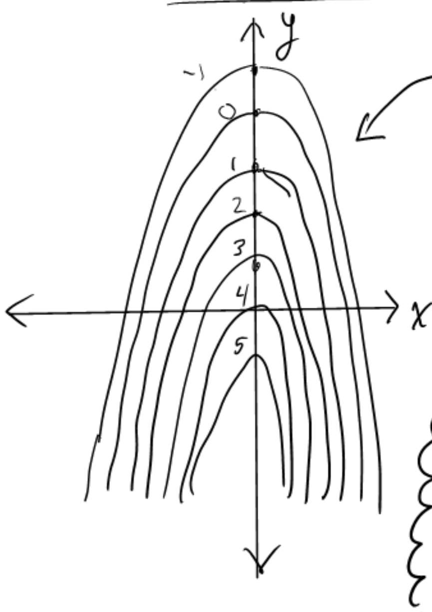
$z = 3$ $y = 1 - x^2$

$z = 2$ $y = 2 - x^2$

$z = 1$ $y = 3 - x^2$

$z = 0$ $y = 4 - x^2$

$z = -1$ $y = 5 - x^2$



Level curves are a "topographical map" of the graph of $z = f(x,y)$

Describes 3-D $(x\ y\ z)$ with 2-D (x, y) diagram

Level Surfaces

Just as level curves describe a 3-D graph of $z=f(x,y)$ in a 2-D drawing, level surfaces can describe a 4-D graph of $w=f(x,y,z)$ in a 3-D drawing.

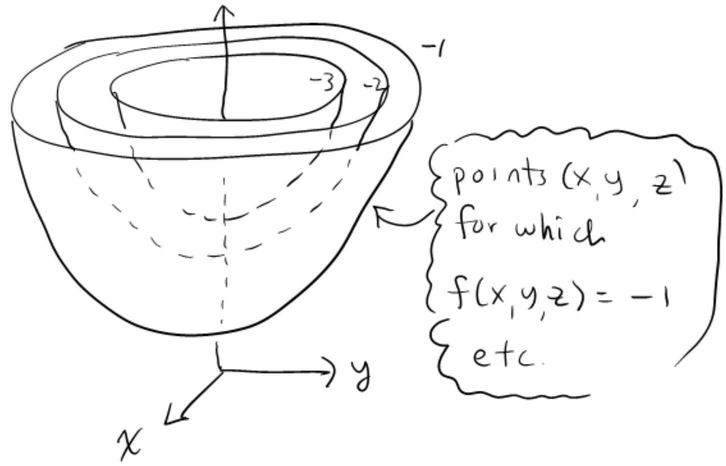
Example Consider

$$w=f(x,y,z)=x^2+y^2-z$$

Level surface for $w=k$

is $k=f(x,y,z)=x^2+y^2-z$

$\rightsquigarrow z=x^2+y^2-k$



Level surface for $k=1$ $\quad z=x^2+y^2-1$

Level surface for $k=0$ $\quad z=x^2+y^2$

Level surface for $k=-1$ $\quad z=x^2+y^2+1$ etc

More on Domains

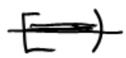
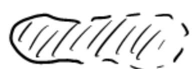
One final thing.

The domain of $f(x)$ tends to be an interval on the x-axis 

the domain of $f(x,y)$ tends to be a region on the xy plane 

Just as intervals can be open, closed or neither, so can regions.

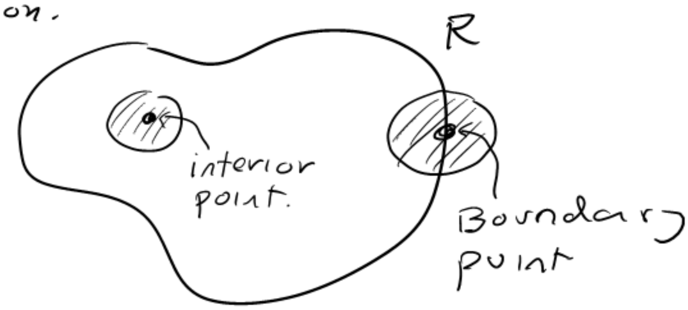
Rough Idea

One Variable	two variables
open interval 	open region 
closed interval 	closed region 
neither open nor closed: 	neither open nor closed 

Precise Definition Suppose R is a region.

A point (a,b) is called an interior point if there is a disk centered at (a,b) that lies entirely inside R.

Point (a,b) is a boundary point if each disk centered at (a,b) contains points both inside and outside R



Region R is open if all points in R are interior points

Region R is closed if it contains all its boundary points

If neither is the case, R is neither open nor closed.