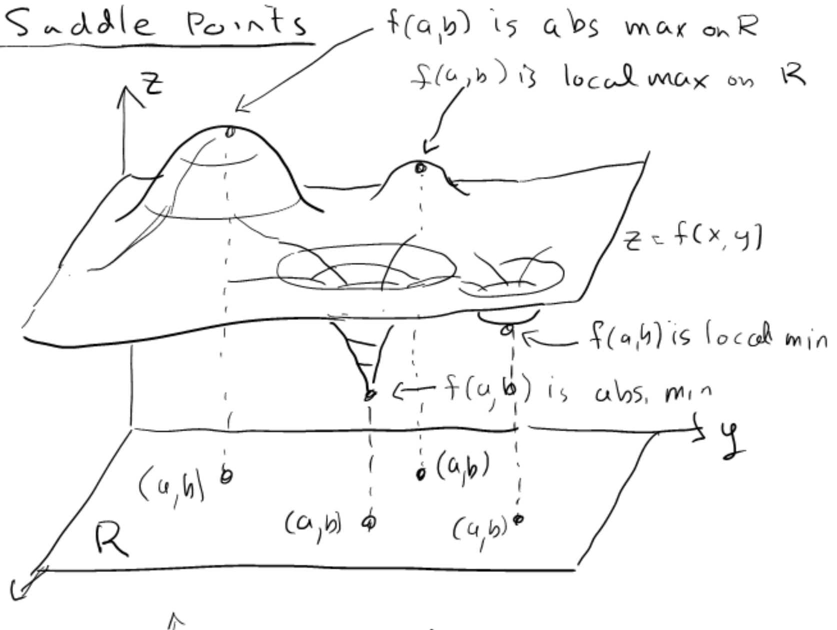
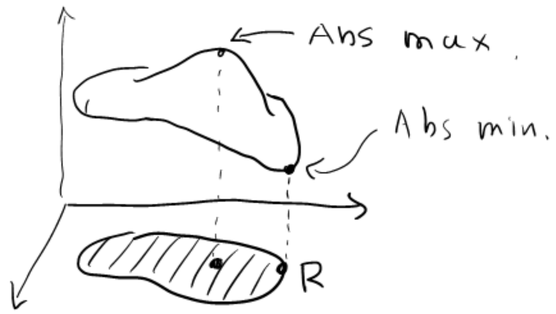
Section 14.7 Extreme Values and Saddle Points

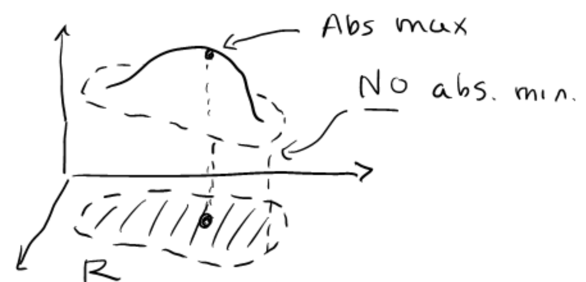


The ideas of local and absolute extrema of $f(x,y)$ carry over from the analogous ideas in one variable. This is illustrated for a $f(x,y)$ defined on a region R

Every absolute maximum (or min) is a local max (or min), but not conversely.



<u>Theorem</u> If $f(x,y)$ is defined on a closed, bounded region R, then it has both an absolute max and an absolute minimum on R, possibly at a boundary point.



But if $f(x,y)$ is defined on an open region R, then it may lack an abs. min or abs max on R.

Absolute max or min values are called <u>absolute extrema</u>
Local max or min values are called <u>local extrema</u>

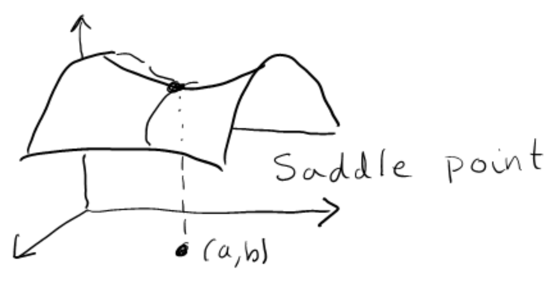
<u>Today's Goal</u> Learn how to find local and absolute extrema

From the pictures above, note that relative extrema occur at points (a,b) for which either $f_x(a,b) = 0 = f_y(a,b)$ <u>or</u> at least one of $f_x(a,b)$ or $f_y(a,h)$ does not exist.

> <u>Definition</u> A point (a,b) in the domain of $f(x,y)$ is called a <u>critical point</u> if either $f_x(a,b) = 0 = f_y(a,b)$ (i.e. $\nabla f(a,b) = \langle 0,0 \rangle$) or at least one of $f_x(a,b)$ and $f_y(a,b)$ does not exist.

Note: Extrema happen at critical points, but not every critical point is the location of an extreme value: Function could be like this:

Critical point (a,b) is a <u>saddle point</u> of $f(x,y)$ if every disk D centered at (a,b), contains points (x,y) with $f(x,y) > f(a,b)$ <u>and</u> points (x,y) with $f(x,y) < f(a,b)$



How to find local extrema

There is no analogue of the first derivative test for local extrema, but there is a second derivative test.

Theorem 11 Second Derivative test

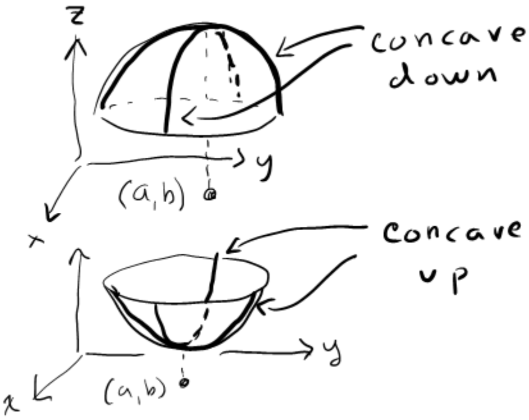
Suppose $f(x,y)$ is defined on an open region containing a critical point (a,b) for which $\nabla f(a,b) = \langle 0,0 \rangle$.
Let $D = f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2$ Then:

1. If $f_{xx}(a,b) < 0$ and $D > 0$, then $f(a,b)$ is a local max.
2. If $f_{xx}(a,b) > 0$ and $D > 0$, then $f(a,b)$ is a local min.
3. If $D < 0$, then there is a saddle point at (a,b)
4. If $D = 0$, there is no conclusion.

Reason:

① $\begin{cases} f_{xx} f_{yy} - f_{xy}^2 > 0 \\ f_{xx} < 0 \Rightarrow f_{yy} < 0 \end{cases}$

② $\begin{cases} f_{xx} f_{yy} - f_{xy}^2 > 0 \\ f_{xx} > 0 \Rightarrow f_{yy} > 0 \end{cases}$



MAX

MIN

Example Find the extrema of $f(x,y) = 2x^4 - x^2 + 3y^2$ on xy-plane.

First find the critical points

$$\nabla f(x,y) = \langle 8x^3 - 2x, 6y \rangle = \langle 2x(4x^2-1), 6y \rangle = \langle 2x(2x-1)(2x+1), 6y \rangle = \langle 0,0 \rangle$$

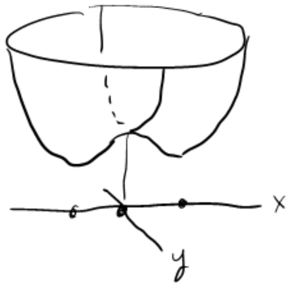
Critical points: $(0,0)$, $(\frac{1}{2},0)$ $(-\frac{1}{2},0)$

Need these $\begin{cases} f_{xx} = 24x^2 - 2 \\ f_{yy} = 6 \\ f_{xy} = 0 \end{cases}$

Point $(0,0)$: $D = f_{xx} f_{yy} - f_{xy} = (24 \cdot 0 - 2) \cdot 6 - 0 < 0$ Saddle point

Point $(\frac{1}{2},0)$ $\left.\begin{array}{l} D = (24(\frac{1}{2})^2 - 2)6 - 0 = 24 > 0 \\ f_{xx}(\frac{1}{2},0) = 4 > 0 \end{array}\right\}$ local min at $(\frac{1}{2},0)$

Point $(\frac{-1}{2},0)$ $\left.\begin{array}{l} D = 24 > 0 \\ f_{xx}(\frac{1}{2},0) = 4 \end{array}\right\}$ local min at $(-\frac{1}{2},0)$



Finding Absolute Extrema on Closed Regions

Recall that absolute extrema of $f(x,y)$ on a closed bounded region are guaranteed to exist, and could occur at a boundary point (even though it may not be a critical point.) and also possibly at critical points.

How to find absolute extrema of $f(x,y)$ on a closed bounded region R:

1. Locate all critical points (a,b) in the interior of R
2. Evaluate $f(a,b)$ for each critical point
3. Investigate behavior of $f(x,y)$ on the boundary.
4. Draw a conclusion from the above information

Investigating the boundary can be tricky at times. In the next section we'll develop a sophisticated method for doing this. However, some situations are relatively easy to deal with.

<u>Example</u> Find the absolute extrema of $f(x,y) = \sin\left(\frac{\pi}{2}(x^2+y^2)\right)$ on the closed disk $R = \{(x,y) \mid x^2+y^2 \le 1\}$

(1) Find critical points

$$\nabla f = \left\langle \cos\left(\frac{\pi}{2}(x^2+y^2)\right)\pi x,\ \cos\left(\frac{\pi}{2}(x^2+y^2)\right)\pi y \right\rangle = \langle 0,0 \rangle$$

<u>Critical points</u> $(0,0)$ and <u>any</u> (a,b) satisfying $a^2+b^2=1$
ie any (a,b) on the boundary.

(2) $f(0,0) = \sin\left(\frac{\pi}{2}(0^2+0^2)\right) = \sin(0) = 0$

(3) Also if (a,b) is on the boundary, i.e. $a^2+b^2=1$, then
$f(a,b) = \sin\left(\frac{\pi}{2}(a^2+b^2)\right) = \sin\frac{\pi}{2} = 1$

(4) From above conclude:

$f(x,y)$ has an <u>absolute minimum</u> of $f(0,0) = 0$ at $(0,0)$

It has an absolute max of 1, occuring at any point (a,b) on the unit circle.