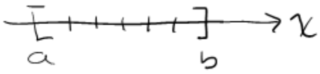
Section 15.5 Triple Integrals

One Variable

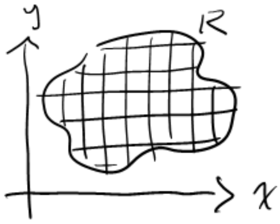
Integrals are over intervals (1-D regions)



$$\int_a^b f(x)\,dx = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

Two Variables

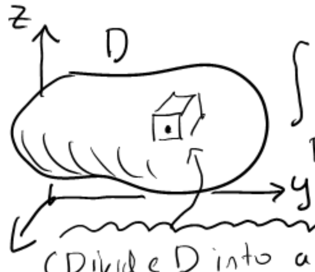
Integrals are over 2-D regions



$$\iint_R f(x,y)\,dA = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k)\underbrace{\Delta x_k \Delta y_k}_{\Delta A_k}$$

Three Variables

Integrals are over 3-D regions



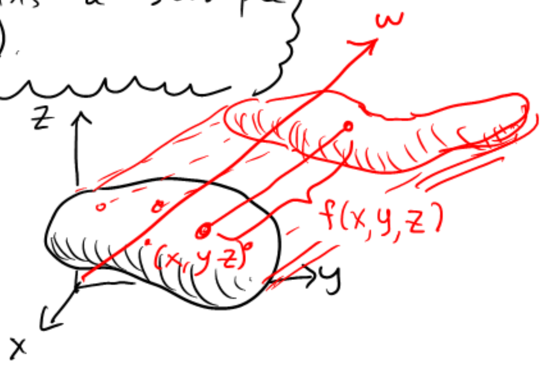
$$\iiint_D f(x,y,z)\,dV = \lim_{|P|\to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\underbrace{\Delta x_k \Delta y_k \Delta z_k}_{\Delta V_k}$$

(Divide D into a grid of n boxes where k^{th} box has dimensions $\Delta x_k \times \Delta y_k \times \Delta z_k$ and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. Also, k^{th} box contains a sample point (x_k, y_k, z_k).)

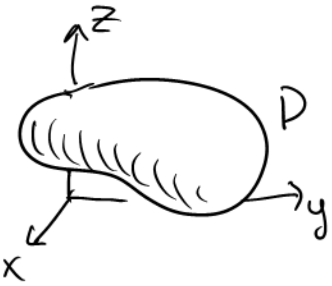
Interpretations

- If $f(x,y,z) \geq 0$ on D then

$$\iiint_D f(x,y,z)\,dV = \left(\begin{array}{l}\text{4-D volume of 4-D} \\ \text{region "below" graph of} \\ w = f(x,y,z) \text{ and "above" } D\end{array}\right)$$



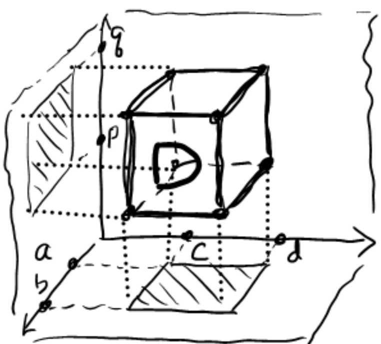
- Volume of D is $V = \iiint_D 1\,dV$



- Average value of $f(x,y,z)$ on D is $\dfrac{\iiint_D f(x,y,z)\,dV}{(\text{volume of } D)}$

Question How to compute $\iiint_D f(x,y,z)\,dV$??

For D = Box, as illustrated

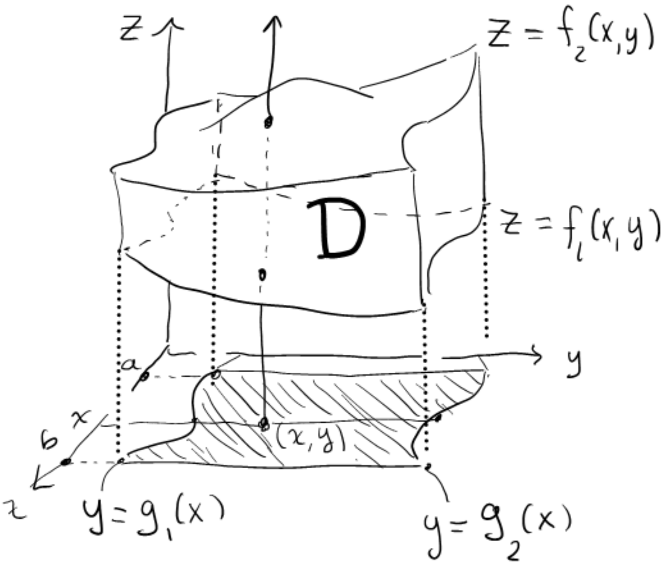


$$\iiint_D f(x,y,z)\,dV = \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx$$

But we can handle more complex regions too!

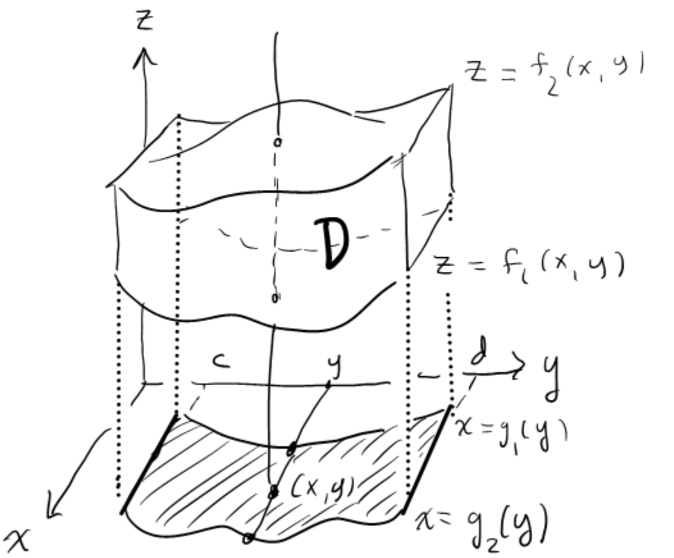
Fubini's Theorem for Triple Integrals

— Requires careful analysis of the solid region D.



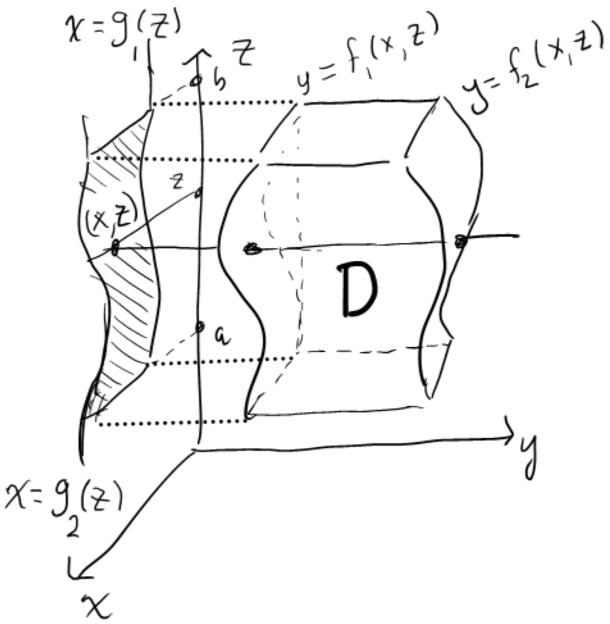
$$\iiint_D F(x,y,z)\,dV =$$

$$\int_a^b \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dz\,dy\,dx$$



$$\iiint_D F(x,y,z)\,dV =$$

$$\int_c^d \int_{x=g_1(y)}^{x=g_2(y)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dz\,dx\,dy$$



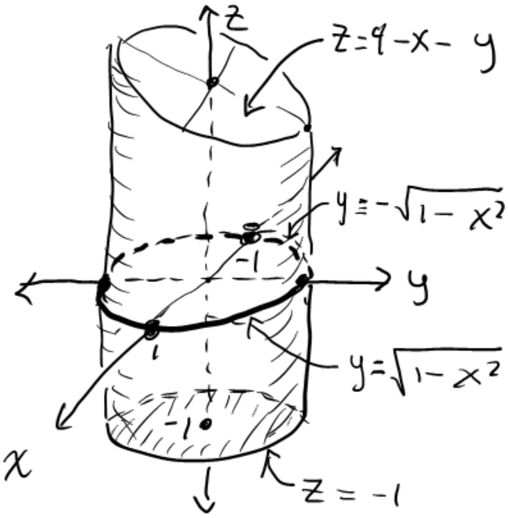
$$\iiint_D F(x,y,z)\,dV =$$

$$\int_a^b \int_{x=g_1(z)}^{x=g_2(z)} \int_{y=f_1(x,z)}^{y=f_2(x,z)} F(x,y,z)\,dy\,dx\,dz$$

It's important to do <u>lots</u> of practice exercises so that you can handle whatever comes your way.

Example Compute $\iiint_D 2xy\,dV$

where D is bounded on sides by $x^2+y^2=1$, on top by $z=4-x-y$ and on bottom by $z=-1$.



$$\iiint_D 2xy\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-1}^{4-x-y} 2xy\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \Big[2xyz\Big]_{-1}^{4-x-y}\,dy\,dx$$

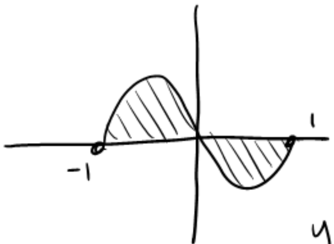
$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2xy(4-x-y)-(2xy(-1))\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 10xy-2x^2y-2xy^2\,dy\,dx$$

$$= \int_{-1}^{1}\Big[5xy^2-x^2y^2-\frac{2}{3}xy^3\Big]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx$$

$$= \int_{-1}^{1}\left(5x(1-x^2)-x^2(1-x^2)-\frac{2}{3}x\sqrt{1-x^2}^{\,3}\right)-\left(5x(1-x^2)-x^2(1-x^2)-\frac{2}{3}x(-\sqrt{1-x^2})^3\right)dx$$

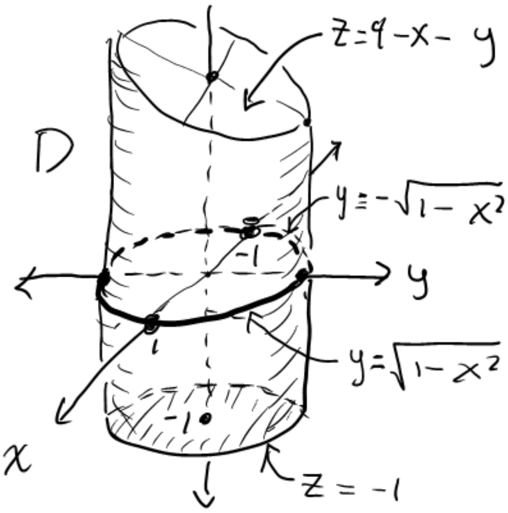
$$= \int_{-1}^{1}-\frac{4}{3}x\sqrt{1-x^2}^{\,3}\,dx = \boxed{0}$$



$y=-\frac{4}{3}x\sqrt{1-x^2}^{\,3}$

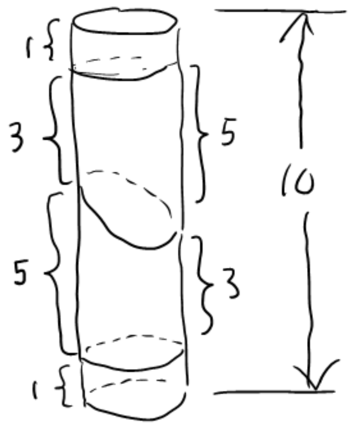
Note: we recognize this last integrand as an odd function, so the integral from -1 to 1 is automatically 0.

Example Now let's compute the volume of the region just considered in the previous problem



Actually the volume can be computed without calculus

Take a copy of D, turn it upside down and place it on top of the first one



Get cylinder of height 10 and Volume $\pi(1)^2 10 = 10\pi$.

Volume of D is twice this

$\boxed{\text{Volume of D is } 5\pi \text{ cubic units}}$

But now lets use calculus and see if we get the same answer

$$\text{Volume} = \iiint_D dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-1}^{4-x-y} dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [z]_{-1}^{4-x-y}\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (5-x-y)\,dy\,dx$$

$$= \int_{-1}^{1}\left[5y - xy - \frac{y^2}{2}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^{1} 10\sqrt{1-x^2} - 2x\sqrt{1-x^2}\,dx$$

$$= \int_{-1}^{1} 10\sqrt{1-x^2}\,dx - \underbrace{\int_{-1}^{1} 2x\sqrt{1-x^2}\,dx}_{\text{zero - odd integrand}} = 10\int_{-1}^{1}\sqrt{1-x^2}\,dx$$

Use trig substitution

$x = \sin\theta$ so

$dx = \cos\theta\,d\theta$

$$= 10\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\;\cos\theta\,d\theta$$

$$= 10\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 10\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= 5\int_{-\pi/2}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta = 5\left[\theta + \tfrac{1}{2}\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{5\pi}$$